PIECEWISE CONSTANT PROFILES MINIMIZING TOTAL VARIATION ENERGIES OF KOBAYASHI-WARREN-CARTER TYPE WITH FIDELITY

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Abstract. We consider a total variation type energy which measures the jump discontinuities different from usual total variation energy. Such a type of energy is obtained as a singular limit of the Kobayashi-Warren-Carter energy with minimization with respect to the order parameter. We consider the Rudin-Osher-Fatemi type energy by replacing relaxation term by this type of total variation energy. We show that all minimizers are piecewise constant if the data is continuous in one-dimensional setting. Moreover, the number of jumps is bounded by an explicit constant involving a constant related to the fidelity. This is quite different from conventional Rudin-Osher-Fatemi energy where a minimizer must have no jump if the data has no jumps. The existence of a minimizer is guaranteed in multi-dimensional setting when the data is bounded.

Contents

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1. Introduction

{SIn}

We consider a kind of total variation energy which measures jumps different from the conventional total variation energy. Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 1$. Let u be in $BV(\Omega)$, i.e., its distributional derivative Du is a finite Radon measure in Ω and let $|Du|$ denote its total variation measure. The total variation energy can be written of the form

$$TV(u) = \int_{\Omega \setminus J_u} |Du| + \int_{J_u} |u^+ - u^-| \, d\mathcal{H}^{n-1},$$

where J_u denotes the (approximate) jump set of u and $u^\pm$ is a trace of u from each side of J_u; $\mathcal{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff measure. For a precise meaning of this formula, see Section 2 and [AFP]. Let $K(\rho)$ be a strictly increasing continuous function for $\rho \geq 0$ with $K(0) = 0$. We set

$$TV_K(u) = \int_{\Omega \setminus J_u} |Du| + \int_{J_u} K\left(|u^+ - u^-|\right) d\mathcal{H}^{n-1}.$$

For a given function $g \in L^2(\Omega)$, we are interested in a minimizer of

$$TV_{Kg}(u) = TV_K(u) + \mathcal{F}(u), \quad \mathcal{F}(u) = \frac{\lambda}{2} \int_\Omega |u - g|^2 \, dx,$$

where $\lambda > 0$ is a constant. The term $\mathcal{F}$ is often called a fidelity term. If $TV_K = TV$, the functional $TV_g(u) = TV(u) + \mathcal{F}(u)$ is often called Rudin-Osher-Fatemi functional since it is proposed by [ROF] to denoise the original image whose grey-level values equal g. For TV_g, there always exists a unique minimizer since the problem is strictly convex and lower semicontinuous in $L^2(\Omega)$. We are interested in regularity of a minimizer of TV_{Kg} assuming some regularity of g. This problem is well studied for TV_g started by [CCN]. Let u_* be the minimizer of TV_g for $g \in BV(\Omega)$. In [CCN], it is shown that $J_{u_*} \subset J_g$ and $u_*^+(x) - u_*^-(x) \leq g^+(x) - g^-(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in J_u$. In particular, if g has no jumps, so does u_*. This type of results is extended in various setting; see a reviewer paper [GKL].

In this paper, we shall show that a minimizer of TV_{Kg} may have jumps even if g has no jumps for some class of subadditive function K including

$$K(\rho) = \frac{\rho}{1+\rho}$$

as a particular example when Ω is an interval. This type of TV_K appears as a kind of singular limit of the Kobayashi-Warren-Carter energy [GOU], [GOSU]. Actually, we shall prove a stronger result saying that a minimizer is a piecewise constant function with finitely many jumps for $g \in C(\overline{\Omega})$ when Ω is a bounded interval. Here is a precise statement.

For a function $K : [0, \infty) \to [0, \infty)$ measuring a jump, we assume that

(K1) K is a continuous, strictly increasing function with $K(0) = 0$.

(K2) For $M > 0$, there exists a positive constant C_M such that

$$K(\rho_1) + K(\rho_2) \geq K(\rho_1 + \rho_2) + C_M \rho_1 \rho_2$$

for all $\rho_1, \rho_2 \geq 0$ with $\rho_1 + \rho_2 \leq M$. In particular, K is subadditive.

(K3) $\lim_{\rho \to 0} K(\rho)/\rho = 1$.

If $K(\rho) = \rho$ so that $TV_K = TV$, K satisfies (K1) and (K3) but does not satisfy (K2). If $K(\rho) = \rho/(1+\rho)$, K satisfies (K2) as well as (K1) and (K3). Indeed, a direct calculation shows that

$$\frac{\rho_1}{1+\rho_1} + \frac{\rho_2}{1+\rho_2} = \frac{\rho + 2\rho_1\rho_2}{1+\rho+\rho_1\rho_2} = \frac{\rho}{1+\rho} + \frac{(2+\rho)\rho_1\rho_2}{(1+\rho)(1+\rho+\rho_1\rho_2)}, \quad \rho = \rho_1 + \rho_2.$$

Thus, (K2) follows.

{TMain}

Theorem 1.1. *Assume that K satisfies (K1), (K2) and (K3). Assume that $g \in C[a,b]$. Let $U \in BV(a,b)$ be a minimizer of TV_{Kg}. Then U must be a piecewise constant function (with finite jumps) satisfying* $\inf g \leq U \leq \sup g$ *on $[a,b]$. Let m be the number of jumps of U. Then*

$$m \leq [(b-a)\lambda/A_M] + 1,$$

where $A_M = \min\{c_M/M, 2C_M\}$ *with*

$$\operatorname{osc}_{[a,b]} g := \max_{[a,b]} g - \min_{[a,b]} g \leq M.$$

Here c_M is a constant such that $K(\rho) \geq c_M \rho$ for $\rho \in [0,M]$. Here $[r]$ denotes the integer part of $r \geq 0$.

We note that we do not assume that $g \in BV(a,b)$. If g is non-decreasing or non-increasing, we have a sharper estimate for m.

{TMainMon}

Theorem 1.2. *Assume that K satisfies (K1), (K2) and (K3). Assume that $g \in C[a,b]$ is non-decreasing (resp. non-increasing). Let $U \in BV(a,b)$ be a minimizer of TV_{Kg}. Then U must be a non-decreasing (non-increasing) piecewise constant function satisfying* $\inf g \leq U \leq \sup g$ *on $[a,b]$. The number m of jumps is estimated as*

$$m \leq [(b-a)\lambda/(2C_M)] + 1$$

for $M \geq \operatorname{osc}_{[a,b]} g = |g(b) - g(a)|$.

To show Theorem 1.1 and also Theorem 1.2, we introduce the notion of a coincidence set C, which is formally defined by

$$C = \left\{ x \in [a,b] \;\middle|\; U(x) = g(x) \right\}.$$

It turns out that a minimizer U of TV_{Kg} is always continuous on C and outside this set C, U is piecewise constant and

$$\sup_{[a,b]} U \leq \sup_{[a,b]} g, \quad \inf_{[a,b]} U \geq \inf_{[a,b]} g. \tag{1.1}$$ {EB0}

Moreover, we are able to prove that U has at most one jump on (α,β) if $(\alpha,\beta) \cap C = \emptyset$ and $\alpha,\beta \in C$. We also prove that if $\alpha,\beta \in C$ with $\alpha < \beta$ is too close, U must be monotone on (α,β). Here, (α,β) may include some point of C. To show these properties, it suffices to assume the subadditivity of K instead of (K2). It includes the case TV_K, where $K(\rho) = \rho$ so that TV_K is the standard total variation.

We shall prove that if $\alpha \in C$ and $\beta \in C$ with $\alpha < \beta$ is too close, U must be a constant on (α,β). A key step (Lemma 4.7) is to show that for a non-decreasing minimizer U on (a,b) and $\alpha,\beta \in C$ with $\alpha < \beta$, the estimate

$$\beta - \alpha > 2C_M/\lambda$$

holds provided that there is $\gamma \in C \cap (\alpha,\beta)$ and $(\gamma,\beta) \cap C = \emptyset$ and that $\rho_2 := U(\beta) - U(\gamma) \geq \rho_1 := U(\gamma) - U(\alpha)$, where C_M is a constant in (K2). Here is a rough strategy to prove the statement. Since $(\gamma,\beta) \cap C = \emptyset$, there is exactly one jump point x_1 in (γ,β). We compare $TV_{Kg}(U)$ and $TV_{Kg}(v)$ in (α,β), where v equals $U(\alpha)$ on (α,x_1) and equals U on (x_1,β). It is not difficult to prove

{EB1} $$TV_K(U) - TV_K(v) \geq K(\rho_1) + K(\rho_2) - K(\rho_1+\rho_2) \geq C_M\rho_1\rho_2 \tag{1.2}$$

since $\rho_1 + \rho_2 \leq M$ by (1.1). The proof for

{EB2} $$\frac{2}{\lambda}\left(\mathcal{F}(U) - \mathcal{F}(v)\right) \geq -\rho_1\rho_2(x_1-\alpha) \tag{1.3}$$

is more involved. It is not difficult to show

{EB3} $$\int_\gamma^{x_1} \left\{(U-g)^2 - (v-g)^2\right\} dx \geq -\rho_1(\rho_1+\rho_2)(x_1-\gamma). \tag{1.4}$$

The proof for

{EB4} $$\int_\alpha^{\gamma} \left\{(U-g)^2 - (v-g)^2\right\} dx \geq -\rho_1^2(\gamma-\alpha) \tag{1.5}$$

is more difficult. If g is non-decreasing, we are able to prove

$$g(x) - U \leq U(x_2+0) - U(x_2-0) \quad \text{for} \quad x \in F,$$

where $F = [x_0,x_2] \subset [a,b)$ is a maximal closed interval (called a facet) such that U is a constant in the interior $\operatorname{int} F$ of F. For general F, one only expects such an inequality just in the average sense, i.e.,

{EB5} $$\int_{x_0}^{x_2} (g(x) - U)\, dx \leq \left(U(x_2+0) - U(x_2-0)\right)(x_2-x_0). \tag{1.6}$$

The estimate (1.5) follows from (1.6). Combining (1.4) and (1.5), we obtain (1.3) since $\rho_2 \geq \rho_1$. The estimates (1.2) and (1.3) yield

$$TV_{Kg}(U) - TV_{Kg}(v) \geq \rho_1\rho_2 \left(C_M - (x_1 - \alpha)\lambda \,/\, 2\right).$$

If $\beta - \alpha \leq 2C_M/\lambda$, U cannot be a minimizer. Similarly, we are able to prove that if U is continuous on (α,β) with $\alpha,\beta \in C$, then U is a constant on (α,β). These observations yield Theorem 1.1. Theorem 1.2 can be proved similarly to Theorem 1.1 by noting that a minimizer U is a monotone function.

We also give a quantative estimate for TV_{Kg} for monotone data $g \in C[a,b]$. It is easy to estimate $TV_{Kg}(u)$ for a piecewise constant u from below. We approximate a general BV function u by piecewise constant functions u_m so that $TV_{Kg}(u_m) \to TV_{Kg}(u)$. Using such an approximation result, we establish an estimate of TV_{Kg} for a general BV function. We notice that such an estimate gives another way to prove Theorem 1.2.

It is not difficult to prove that TV_{Kg} is lower semicontinuous in the space of piecewise constant functions with at most k jumps. Since the space is of finite dimension, its bounded closed set is compact. By Weierstrass' theorem, TV_{Kg} admits a minimizer v among piecewise constant functions with at most k jumps with $\inf g \leq v \leq \sup g$. Thus, Theorem 1.1 guarantees the existence of a minimizer in $BV(a,b)$. The existence of minimizer of TV_{Kg} itself can be proved for general essential bounded measurable function g, i.e., $g \in L^\infty(\Omega)$ for general Lipschitz domain in $\mathbb{R}^n$ since it is known [AFP] that TV_K is lower semicontinuous in a suitable topology. We shall discuss this point in Section 2. Note that instead of (K2) subadditivity for K is enough to have the existence of a minimizer.

We believe Theorem 1.1 extends for general $g \in L^\infty(a,b)$. In fact, we have a weaker version.

{TGen}

Theorem 1.3. *Assume that K satisfies (K1), (K2) and (K3). For* $g \in L^\infty(a,b)$*, there exists a piecewise constant function U having at most*

$$\left[(b-a)\lambda \,/\, A_M\right] + 1$$

jumps and satisfying $\operatorname{ess.inf} g \leq U \leq \operatorname{ess.sup} g$ *on* (a,b)*, which minimizes* TV_{Kg} *on* $BV(a,b)$*. Here,* $M \geq \operatorname{osc}_{[a,b]} g = \operatorname{ess.sup} g - \operatorname{ess.inf} g$.

This follows from above-mentioned approximation result and Theorem 1.1. Since the uniqueness of minimizer is not guaranteed in general, there might exist another minimizer which is not piecewise constant although it is unlikely.

As shown in [GOU], [GOSU], TV_{K_g} is obtained as a singular limit ($\varepsilon \downarrow 0$ limit) of the Kobayashi-Warren-Carter type energy [KWC1, KWC2, WKC]

$$E^{\varepsilon}_{\mathrm{KWC}_g}(u,v) := E^{\varepsilon}_{\mathrm{KWC}}(u,v) + \mathcal{F}(u)$$
$$E^{\varepsilon}_{\mathrm{KWC}}(u,v) := \int_\Omega sv^2|Du| + E^{\varepsilon}_{\mathrm{sMM}}(v), \quad s>0$$
$$E^{\varepsilon}_{\mathrm{sMM}}(v) := \frac{\varepsilon}{2}\int_\Omega |\nabla v|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega F(v)\,dx$$

by minimizing order parameter v; here $F(v)$ is a single-well potential typically of the form $F(v) = (v-1)^2$ and s is a positive parameter. In fact, in one-dimensional setting [GOU], the Gamma limit of $E^{\varepsilon}_{\mathrm{sMM}}(v)$ under the graph convergence formally equals

$$E^{0}_{\mathrm{sMM}}(\Xi) = \sum_{i=1}^{\infty} 2\left(G(\xi_i^- + G(\xi_i^+)\right), \quad G(s) = \left|\int_1^s \sqrt{F(\xi)}\,d\xi\right|$$

if the limit of v in $\Omega = (a,b)$ equals a set-valued function Ξ of the form

$$\Xi(x) = \begin{cases} 1, & x \notin \Sigma, \\ [\xi_i^-, \xi_i^+](\ni 1) & \text{for} \quad x \in \Sigma, \end{cases}$$

where Σ is some (at most) countable set. The Gamma limit of $E^{\varepsilon}_{\mathrm{KWC}}$ equals

$$E^{0}_{\mathrm{KWC}}(u,\Xi) = \sum_{i=1}^{\infty}\left(s(\xi^-_{i,+})^2\left|u^+(x_i) - u^-(x_i)\right| + 2\left(G(\xi_i^-) + G(\xi_i^+)\right)\right) + \int_{\Omega\setminus J_u}|Du|,$$

where $\xi_+ = \max(\xi, 0)$ and $x_i \in \Sigma$. Here for u, L^1 type limit is considered. If we minimize $E^{0}_{\mathrm{KWC}}(u,\Xi)$ by fixing u, ξ^+ must be one since $[\xi^-,\xi^+] \ni 1$ and $G(1) = 0$. Thus

$$\begin{aligned}\inf_{\Xi} E^{0}_{\mathrm{KWC}}(u,\Xi) &= \sum_{i=1}^{\infty}\min_{\xi>0}\left(s(\xi_+)^2\left|u^+(x_i) - u^-(x_i)\right| + 2G(\xi)\right) + \int_{\Omega\setminus J_u}|Du| \\ &= TV_K(u) + \int_{\Omega\setminus J_u}|Du|,\end{aligned}$$

where

{EKdef}
$$K(\rho) = \min_{\xi}\left(s(\xi_+)^2\rho + 2G(\xi)\right). \tag{1.7}$$

In the case, $s = 1$ and $F(v) = (v-1)^2$, a direct calculation shows that

$$K(\rho) = \min_{\xi>0}\left(\xi^2\rho + (\xi-1)^2\right) = \frac{\rho}{\rho+1}.$$

We shall prove that K defined in (1.7) satisfies (K2) provided that

$$\overline{\lim_{v\uparrow 1}} F'(v) \,/\, (v-1) < \infty$$

for F if $F \in C(\mathbb{R})$ is non-negative and $F(v) = 0$ if and only if $v = 0$.

If we replace $\int sv^2|Du|$ by $\int sv^2|Du|^2$, the energy corresponding to $E^{\varepsilon}_{\mathrm{KWC}g}$ is nothing but what is called the Ambrosio-Tortorelli energy [AT]. Its singular limit is a Mumford-Shah functional

$$E_{\mathrm{MS}}(u, K) := s \int_{\Omega \setminus K} |\nabla u|^2 + \mathcal{H}^{n-1}(K) + \mathcal{F}(u),$$

where K is a closed set in Ω [AT], [AT2], [FL]. The existence of a minimizer is obtained in [GCL] by using the space of special functions with bounded variation, i.e., SBV functions which is a subspace of $BV(\Omega)$.

A modified total variation energy TV_K is not limited to a singular limit of the Kobayashi-Warren-Carter energy. In fact, $TV_K(u)$ like energy is derived as the surface tension of grain boundaries in polycrystals by [LL], where u is taken as a piecewise constant (vector-valued) function; see also [GaFSp] for more recent development. The function K measuring jumps may not be isotropic but still concave. In [ELM], TV_K type energy for a piecewise constant function is also considered to study motion of a grain boundary. However, in their analysis, the convexity of K is assumed.

This paper is organized as follows. In Section 2, we give a rigorous formulation of TV_K and prove the existence of a minimizer of TV_{Kg} for $g \in L^\infty(\Omega)$ for a general Lipschitz domain in $\mathbb{R}^n$. In Section 3, we study a profile of a minimizer U for TV_{Kg} including TV_g outside the coincidence set. We also prove that U is monotone in (α, β) with $\alpha, \beta \in C$ provided that α and β is close. In Section 4, we prove that $\alpha, \beta \in C$ cannot be too close if K satisfies (K2). We prove Theorem 1.1, Theorem 1.2 and Theorem 1.3. We also establish an estimate for TV_{Kg} for monotone g. In Section 5, we shall discuss a sufficient condition that K in (1.7) satisfies (K2).

2. Existence of a minimizer {SEx}

In this section, after giving a precise definition of TV_K, we give an existence result for its Rudin-Osher-Fatemi type energy. The proof is based on a standard compactness result for TV and a classical lower semicontinuity result for TV_K.

We recall a standard notation as in [AFP]. Let Ω be a domain in $\mathbb{R}^n$. For a locally integrable (real-valued) function u, we consider its total variation $TV(u)$ in Ω is defined as

$$TV(u) = \sup\left\{ \int_\Omega -u \operatorname{div} \varphi \, dx \;\middle|\; \varphi \in C_c^\infty(\Omega, \mathbb{R}^n),\ |\varphi(x)| \leq 1 \text{ in } \Omega \right\},$$

where $C_c^\infty(\Omega, \mathbb{R}^n)$ denotes the space of all $\mathbb{R}^n$-valued smooth functions with compact support in Ω. An integrable function u, i.e., $u \in L^1(\Omega)$, is called a function of bounded variation if $TV(u) < \infty$. The space of all such function is denoted by $BV(\Omega)$, i.e.,

$$BV(\Omega) = \left\{ u \in L^1(\Omega) \;\middle|\; TV(u) < \infty \right\}.$$

By Riesz's representation theory, one easily observe that $TV(u)$ is finite if and only if the distributional derivative Du of u is a finite Radon measure on Ω and its total variation $|Du|(\Omega)$ in Ω equals $TV(u)$.

We next define a jump discontinuity of a locally integrable function. Let $B_r(x)$ denote an open ball of radius r centered at x in $\mathbb{R}^n$. In other words,

$$B_r(x) = \left\{ y \in \mathbb{R}^n \;\middle|\; |y - x| < r \right\}.$$

For a unit vector $\nu \in \mathbb{R}^n$, we define a half ball of the form

$$B_r^{\pm}(x, \nu) = \left\{ y \in B_r(x) \;\middle|\; \pm\nu \cdot (y - x) \geq 0 \right\},$$

where $a \cdot b$ for $a, b \in \mathbb{R}^n$ denotes the standard inner product in $\mathbb{R}^n$. Let w be a locally integrable function in Ω. We say that a point $x \in \Omega$ is a (approximate) jump point of w if there exists a unit vector $\nu_w \in \mathbb{R}^n$, $w^{\pm} \in \mathbb{R}$, $w^+ \neq w^-$, such that

$$\lim_{r \downarrow 0} \frac{1}{\mathcal{L}^n \left(B_r^{\pm}(x, \nu_w) \right)} \int_{B_r^{\pm}(x,\nu_w)} \left| w(y) - w^{\pm} \right| dy = 0.$$

Here $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$ so this integral is the average of $|w(y) - w^{\pm}|$ over $B_r^{\pm}(x, \nu_w)$. The set of all jump points of w is denoted by J_w and called the (approximate) jump (set) of w. By definition, J_w is contained in the set S_w of (approximate) discontinuity point of w, i.e.,

$$S_w = \left\{ x \in \Omega \;\middle|\; \lim_{r \downarrow 0} \frac{1}{\mathcal{L}^n \left(B_r(x) \right)} \int_{B_r(x)} |w(y) - z| \, dt = 0 \text{ does not hold} \right. \\ \left. \text{for any choice of } z \in \mathbb{R} \right\}.$$

By the Federer-Vol'pert theorem [AFP, Theorem 3.78], we know $\mathcal{H}^{n-1}$ $(S_u \backslash J_u) = 0$, where $\mathcal{H}^m$ denotes m-dimensional Hausdorff measure provided that $u \in BV(\Omega)$. Moreover, S_u is countably $\mathcal{H}^{n-1}$-rectifiable, i.e., S_u can be covered by a countable union of Lipschitz graphs up to an $\mathcal{H}^{n-1}$ measure zero set. In particular, J_u is also countably $\mathcal{H}^{n-1}$-rectifiable. Quite recently, it is proved that J_u is always countably $\mathcal{H}^{n-1}$-rectifiable if we only assume that w is locally integrable [DN]. For $u \in BV(\Omega)$, the value $u^{\pm}$ can be viewed as a trace of u on a countably $\mathcal{H}^{n-1}$-rectifiable set [AFP, Theorem 3.77, Remark 3.79] except $\mathcal{H}^{n-1}$ negligible set (up to permutation of u^+ and u^-). We now recall a unique decomposition of the Radon measure Du for $u \in BV(\Omega)$ of the form

$$Du = D^a u + D^s u, \quad D^s u = D^c u + (u^+ - u^-) \cdot \nu_u \mathcal{H}^{n-1} \lfloor J_u;$$

see [AFP, Section 3.8]. The term $D^a u$ denotes the absolutely continuous part and $D^s u$ denotes the singular part with respect to the Lebesgue measure. The term $D^a u = \nabla u \mathcal{L}^n$, where $\nabla u \in \left(L^1(\Omega)\right)^n$. The singular part is decomposed into two parts; $D^c u$ vanishes on sets of finite $\mathcal{H}^{n-1}$ measure. For a measure μ on Ω and a set $A \subset \Omega$, the associate measure $\mu \lfloor A$ is defined as

$$(\mu \lfloor A)(W) = \mu(A \cap W), \quad W \subset \Omega.$$

We now consider a total variation type energy measuring jumps in a different way. For $u \in BV(\Omega)$, we set

$$TV_K(u) := (TV_K(u, \Omega) :=) \int_{\Omega \setminus J_u} |Du| + \int_{J_u} K\left(|u^+ - u^-|\right) d\mathcal{H}^{n-1}.$$

Here the density function is assumed to satisfy following conditions.

(K1w) $K : (0, \infty) \to [0, \infty)$ is non-decreasing, lower semicontinuous.
(K2w) K is subadditive, i.e., $K(\rho_1 + \rho_2) \leq K(\rho_1) + K(\rho_2)$.
(K3) $\lim_{\rho \to 0} K(\rho)/\rho = 1$.

{TEx1}

Theorem 2.1. *Assume that K satisfies (K1w), (K2w) and (K3). Let Ω be a bounded domain with Lipschitz boundary in $\mathbb{R}^n$. Let $\mathcal{E}$ be a lower semicontinuous function in $L^1(\Omega)$ with values in $[0, \infty]$. Then $TV_K + \mathcal{E}$ has a minimizer on $BV(\Omega)$ provided that a coercivity condition*

$$\inf_{u \in BV(\Omega)} (TV_K + \mathcal{E})(u) = \inf_{\substack{u \in BV(\Omega) \\ \|u\|_\infty \leq M}} (TV_K + \mathcal{E})(u)$$

for some $M > 0$, where $\|\cdot\|_\infty$ denotes the L^∞-norm.

We consider the Rudin-Osher-Fatemi type energy for TV_K, i.e., for $g \in L^2(\Omega)$,

$$TV_{Kg}(u) := TV_K(u) + \mathcal{F}(u), \quad \mathcal{F}(u) = \frac{\lambda}{2} \int_\Omega |u - g|^2 \, dx.$$

{TEx2}

Theorem 2.2. *Assume that K satisfies (K1w), (K2w) and (K3). Let Ω be a bounded domain with Lipschitz boundary in $\mathbb{R}^n$. Assume that $g \in L^\infty(\Omega)$. Then there is an element $u_0 \in BV(\Omega)$ such that*

$$TV_{Kg}(u_0) = \inf_{u \in BV(\Omega)} TV_{Kg}(u).$$

In other words, there is at least one minimizer of TV_{Kg}.

Proof of Theorem 2.2 admitting Threorem 2.1. In the case $\mathcal{E} = \mathcal{F}$, the lower semicontinuity of $\mathcal{E}$ in $L^1(\Omega)$ is rather clear. If $g \in L^\infty(\Omega)$, then for a chopped function $u_M = \max(\min(u, M), -M)$ with $M > ||g||_\infty$, we see that

$$\frac{2}{\lambda}(\mathcal{F}(u) - \mathcal{F}(u_M)) = \int_{u\geq M} |u-g|^2\,dx - \int_{u\geq M} |M-g|^2\,dx$$
$$+ \int_{u\leq -M} |u-g|^2\,dx - \int_{u\leq -M} |-M-g|^2\,dx \geq 0.$$

Thus the coercivity condition is fulfilled, and Theorem 2.2 now follows from Theorem 2.1. □

In the rest of this section, we shall prove Theorem 2.1 by a simple direct method. We begin with compactness.

{PCom}

Proposition 2.3. *Assume that (K1w) and (K3) are fulfilled. Assume that Ω is a bounded domain with Lipschitz boundary in $\mathbb{R}^n$. Let $\{u_k\}_{k=1}^\infty$ be a sequence in $BV(\Omega)$ such that*

$$\sup_{k\geq 1} TV_K(u_k) < \infty \quad \textit{and} \quad \sup_{k\geq 1} ||u_k||_\infty < \infty.$$

Then there is a subsequence $\{u_{k'}\}$ and $u \in BV(\Omega)$ such that $u_{k'} \to u$ strongly in $L^1(\Omega)$ and $Du_{k'} \to Du$ weak in the space of bounded measures. In other words, $u_{k'}$ is sequentially weakly* converges to u in $BV(\Omega)$.*

Proof. By (K1) and (K3), we see that for any M, there is $c'_M > 0$ such that

{EKEB}
$$K(\rho) \geq c'_M \rho \quad \text{for} \quad \rho \leq M. \tag{2.1}$$

If M is chosen such that $||u_k||_\infty \leq M$, then

$$TV(u_k) \leq \frac{1}{c'_M} TV_K(u_k).$$

Thus $\{TV(u_k)\}$ is bounded. By the standard compactness for $BV(\Omega)$ function [AFP, Theorem 3.23], [Giu, Theorem 1.19] yields the desired results. □

For a lower semicontinuity, we have

{PLS}

Proposition 2.4. *Assume (K1w), (K2w) and (K3), then TV_K is sequentially weakly* lower semicontinuous in $BV(\Omega)$.*

This is a special form of the lower semicontinuity result [AFP, Theorem 5.4]. We restate it for the reader's convenience. We consider

$$F(u) = \int_{\Omega\setminus J_u} \varphi(|\nabla u|)\,dx + \beta|D^c u|(\Omega) + \int_{J_u} K(|u^+ - u^-|)\,d\mathcal{H}^{n-1}.$$

{PLS2}

Proposition 2.5. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a non-decreasing, lower semicontinuous and convex function. Assume that $K : (0,\infty) \to [0,\infty)$ is a non-decreasing, lower semicontinuous and subadditive function and $\beta \in [0,\infty)$.*

Then F is sequentially weakly lower semicontinuous in $BV(\Omega)$ provided that*

$$\lim_{t\uparrow\infty} \frac{\varphi(t)}{t} = \beta = \lim_{t\downarrow 0} \frac{K(t)}{t}.$$

See [AFP, Theorem 5.4]. Such a lower semicontinuity is originally due to Bouchitté and Buttazzo [BB]. In our setting $\varphi(t) = t, \beta = 1$.

Proof of Theorem 2.1. Let $\{u_k\}_{k=1}^{\infty}$ be a minimizing sequence of $TV_K + \mathcal{E}$, i.e.,

$$\lim_{k\to\infty} (TV_K + \mathcal{E})(u_k) = \inf_{u\in BV} (TV_K + \mathcal{E}).$$

By compactness (Proposition 2.3), $\{u_k\}$ contains a convergent subsequence still denoted by $\{u_k\}$ to some $u \in BV(\Omega)$, sequentially weakly* in $BV(\Omega)$. By lower semicontinuity of TV_K (Proposition 2.4), we conclude that

$$(TV_K + \mathcal{E})(u) \leq \varliminf_{k\to\infty} (TV_K + \mathcal{E})(u_k).$$

Thus, u is a minimizer of $TV_K + \mathcal{E}$ in $BV(\Omega)$. □

3. Coincidence set of a minimizer {SCoi}

In this section, we discuss one-dimensional setting and study properties of coincidence set

$$C = \left\{ x \in \Omega \;\middle|\; U(x) = g(x) \right\}$$

of a minimizer U.

Let Ω be a bounded open interval, i.e., $\Omega = (a, b)$. We consider $TV_{Kg}(u)$ for $g \in C[a, b]$ for $u \in BV(a, b)$. Since u can be written as a difference of two non-decreasing function, we may assume that u has a representative that J_u is at most a countable set and outside J_u, u is continuous. Moreover, we may assume that u is right (resp. left) continuous at a (resp. at b). For $x \in J_u$, $u(x \pm 0)$ is well-defined.

For K, we assume

(K1) $K : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing function with $K(0) = 0$.

We first prove that a minimizer U is piecewise constant in the place $U > g$ or $U < g$.

{LPic}

Lemma 3.1. *Assume that K satisfies (K1) and that $g \in C[a, b]$. Let $U \in BV(a, b)$ be a minimizer of TV_{Kg}. Let $x_0 \in (a, b)$ be a continuous point of U and assume $U(x_0) > g(x_0)$ (resp. $U(x_0) < g(x_0)$). Then U is constant in some interval (α, β) including x_0 and $U(x) > g(x)$ (resp. $U(x) < g(x)$) for $x \in (\alpha, \beta)$. Moreover, we can take (α, β) such that one of following three cases occurs exclusively.*

(i) $U(\alpha - 0) < g(\alpha) < U(\alpha + 0)$ *(resp.* $U(\alpha - 0) > g(\alpha) > U(\alpha + 0)$*) and* $U(\beta - 0) = g(\beta)$*,*

(ii) $U(\alpha + 0) = g(\alpha)$ *and* $U(\beta - 0) > g(\beta) > U(\beta + 0)$ *(resp.* $U(\beta - 0) < g(\beta) < U(\beta + 0)$*),*

(iii) $U(\alpha + 0) = g(\alpha)$ *and* $U(\beta - 0) = g(\beta)$*.*

In the case $\alpha = a$*, we do not impose the condition* $g(\alpha) > U(\alpha - 0)$ *(resp.* $g(\alpha) < U(\alpha - 0)$*) and similarly,* $g(\beta) > U(\beta + 0)$ *(resp.* $g(\beta) < U(\beta + 0)$*) is not imposed for* $\beta = b$ *since* $U(\alpha - 0)$*,* $U(\beta + 0)$ *are undefined.*

Proof. We shall only give a proof for the case $U(x_0) > g(x_0)$ since the argument for $U(x_0) < g(x_0)$ is symmetric. We consider the case that U is continuous on (α,β) with $U > g$ on (α,β) and $U(\alpha + 0) = g(\alpha)$, $U(\beta - 0) = g(\beta)$. We shall claim that U is a constant. If not, $\max_{[\alpha,\beta]} U > \min_{[\alpha,\beta]} U$. There would exist two points x_1 and x_2 in $[\alpha,\beta]$ such that $U(x_1) = \min_{[\alpha,\beta]} U$ and $U(x_2) = \max_{[\alpha,\beta]} U$ such that $U(x) \in (\min U, \max U)$ for x between x_1 and x_2. We may assume $x_1 < x_2$ since the proof for the other case is symmetric. We set

$$v(x) = \begin{cases} \tilde{U}(x), & x \in (x_1, x_2), \\ U(x), & x \notin (x_1, x_2), \end{cases}$$

where $\tilde{U}$ is a continuous non-decreasing function on $[x_1, x_2]$ such that $\tilde{U}(x) < U(x)$ for $x \in (x_1, x_2)$ and $U(x_1) = \tilde{U}(x_1)$, $U(x_2) = \tilde{U}(x_2)$. Then $\mathcal{F}(v) < \mathcal{F}(U)$ and $TV_K(v) = TV(v) \leq TV(U) = TV_K(U)$. This would contradict to the assumption that U is a minimizer of TV_{Kg}. Hence, U is a constant and case (iii) occurs.

Assume that there is a jump point α_0 of U with $\alpha_0 < x_0$ with $U(\alpha_0 + 0) > g(\alpha_0)$. Then $U(\alpha_0 + 0) > U(\alpha_0 - 0)$. If not, $d = U(\alpha_0 - 0) - U(\alpha_0 + 0) > 0$ and set

$$v(x) = \begin{cases} U(x) - d, & x \in (\alpha_0 - \delta, \alpha_0) \\ U(x), & x \notin (\alpha_0 - \delta, \alpha_0). \end{cases}$$

For a sufficiently small $\delta > 0$, $v(x) > g(x)$; see Figure 1. By definition, $TV_K(v) \leq TV_K(U)$ and $\mathcal{F}(v) < \mathcal{F}(U)$. Thus U is not a minimizer of TV_{Kg}. Similarly, if there is a jump point β_0 of U with $x_0 < \beta_0$, then $U(\beta_0 - 0) > U(\beta_0 + 0)$.

We shall prove that U is continuous on (x_0,β) provided that $U > g$ on (α_0,β). If not, there would exist a jump point β_0 of U with $\alpha_0 < x_0 < \beta_0 < \beta$ satisfying $U(x) > g(x)$ for $x \in (\alpha_0,\beta_0)$, $U(\alpha_0+0) > g(\alpha_0)$, $U(\beta_0-0) > g(\beta_0)$. We shall prove that such configuration does not occur. We first claim that $U(\alpha_0 + 0) = U(\beta_0 - 0)$. By symmetry, we may assume that $U(\alpha_0 + 0) < U(\beta_0 - 0)$. Since we know $U(\beta_0 + 0) < U(\beta_0 - 0)$, we take M such that

$$\max\left(U(\alpha_0 + 0), U(\beta_0 + 0)\right) < M < U(\beta_0 - 0).$$

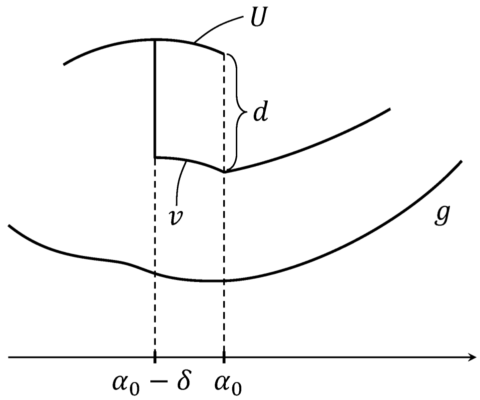


FIGURE 1. shift of a jump Fshi

We set $u_M = \min(M, U)$ on (α_0, β_0). Outside (α_0, β_0), we set $u_M = U$. If M is taken close to $U(\beta_0 - 0)$, then $u_M > g$ on (α_0, β_0) and $u_M(\alpha_0 + 0) > g(\alpha_0)$, $u_M(\beta_0 - 0) > g(\beta_0)$; see Figure 2. Thus $\mathcal{F}(u_M) \leq \mathcal{F}(U)$ and by

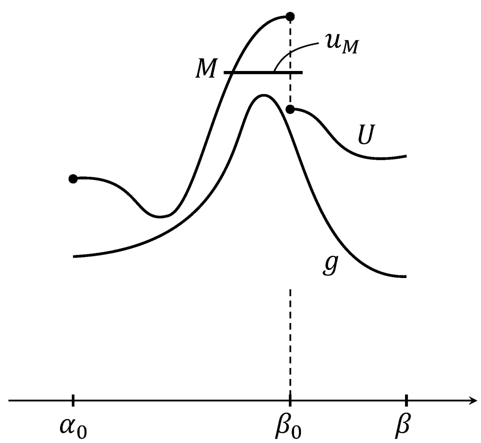


FIGURE 2. a way of truncation Ftrun

truncation $TV_K(u_M) < TV_K(U)$. This contradicts to the assumption that U is a minimizer. Thus $U(\alpha_0 + 0) = U(\beta_0 - 0)$. However, if so, take M again close to $U(\alpha_0 + 0)$ and $M < U(\alpha_0 + 0)$, $TV_K(u_M) < TV_K(U)$ with $\mathcal{F}(u_M) \leq \mathcal{F}(U)$. This is a contradiction so if α_0 is a jump of U for $\alpha_0 < x_0$, then there is no jump on (x_0, β) provided that $U(x) > g(x)$. In other words, U is continuous and satisfies $U(x) > g(x)$ on $[x_0, \beta)$ and $U(\beta - 0) = g(\beta)$. We now conclude that U is constant on $[x_0, \beta)$ by using a similar argument at the beginning of this proof.

Since there is a sequence of continuity point x_j of U converging to α_0 as $j \to \infty$ keeping $x_j > \alpha_0$, we conclude that U is a constant on (x_j, β).

This says that U is constant on (α_0, β). To say that this corresponds to (i), it remains to prove that $U(\alpha - 0) < g(\alpha)$. If not, $U(\alpha - 0) \geq g(\alpha)$, then we take

$$v(x) = \begin{cases} \max\left(U(x) - d, \tilde{g}(x)\right), & x \in (\alpha_0, \alpha_0 + \delta), \\ U(x), & x \notin (\alpha_0, \alpha_0 + \delta), \\ U(x - 0), & x = \alpha_0, \end{cases}$$

where $d = U(\alpha_0 + 0) - U(\alpha - 0)$. Here,

$$\tilde{g}(x) = \sup\left\{ g(y) - g(\alpha_0) \,\middle|\, \alpha_0 \leq y \leq x \right\} + g(\alpha_0)$$

which is continuous and non-decreasing. Moreover,

$$TV_K(v) \leq TV_K(U) + m\left(g(\delta)\right), \quad \mathcal{F}(v) \leq \mathcal{F}(U) - \delta\left(d - \tilde{g}(\delta)\right)^2,$$

where $m(\sigma) = \sup\left\{ K(d + \tau) - K(d) \,\middle|\, 0 \leq \tau \leq \sigma \right\}$; see Figure 3. Thus

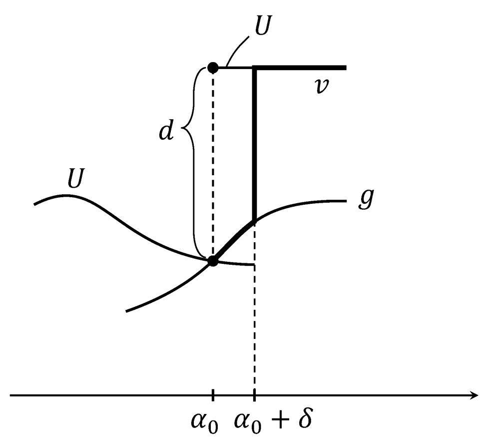


FIGURE 3. modification of U Fmod

$$TV_{Kg}(v) \leq TV_{Kg}(U) + m\left(\tilde{g}(\delta)\right) - \delta\left(d - \tilde{g}(\delta)\right)^2.$$

Since $\tilde{g}(\delta) \to 0$ as $\delta \to 0$, and $m(\sigma) \to 0$ as $\sigma \to 0$ by (K1), we conclude that for sufficiently small $\delta > 0$, $TV_{Kg}(v) < TV_{Kg}(U)$. We now obtain (i). A symmetric argument yields (ii). The proof is now complete. □

For $u \in BV(a, b)$ and $g \in C[a, b]$, we set

$$C_\pm = \left\{ x \in [a, b] \,\middle|\, u(x \pm 0) = g(x) \right\}.$$

If $C_+ = C_-$ in (a, b), we simply write

$$C = C_+ \cup C_-$$

and call C the *coincidence set* of u. By Lemma 3.1, we obtain a few properties of C.

{LCoi}

Lemma 3.2. *Assume the same hypotheses of Lemma 3.1 concerning K, g and U. Let $C_\pm$ be defined for $u = U$. Then $C_- = C_+$ in (a,b), i.e., U is continuous on the coincidence set C. If $(x_1, x_2) \cap C = \emptyset$ with $x_1, x_2 \in C$. then U is piecewise constant on (x_1, x_2) with at most one jump.*

Proof. By Lemma 3.1 (i) and (ii), we easily see that $J_U \cap C_\pm = \emptyset$. Thus, U is continuous on C.

It remains to prove the second statement. By Lemma 3.1, the value of U on (x_1, x_2) is either $g(x_1)$ or $g(x_2)$. Moreover, if there are more than two jumps, U must take value of g at some point $x_* \in (x_1, x_2)$. In other words, $x_* \in C$. This contradicts to $C \cap (x_1, x_2) = \emptyset$. □

We conclude this section by showing that if two points $\alpha, \beta \in C$ with $\alpha < \beta$ for a minimizer U is too close, then U must be monotone in (α, β) under the assumption that K satisfies (K1), (K2w) and (K3).

We first note comparison with usual total variation TV and TV_K.

{LCpTK}

Lemma 3.3. *Assume that K satisfies (K1), (K2w) and (K3). If $u \in BV(\Omega)$ with $\Omega = (a,b)$ is continuous at a and b, then*

$$TV_K(u) \geq K(\rho) \quad \textit{with} \quad \rho = |u(b) - u(a)|.$$

Proof. We may assume that $u(a) < u(b)$. By definition,

$$TV_K(u) = \int_{\Omega \setminus J_u} |Du| + \sum_{x_i \in J_u} K(\rho_i)$$

where J_u denote the jump discontinuity of u and $\rho_i = |u(x_i + 0) - u(x_i - 0)|$ for $x_i \in J_u$. We note that

$$\int_{\Omega \setminus J_u} |Du| \geq \left(\rho - \sum_{x_i \in J_u^+} \rho_i \right)_+,$$

where $J_u^+ = \left\{ x \in J_u \;\middle|\; u(x+0) - u(x-0) > 0 \right\}$. By subadditivity (K2w), we see that

$$\sum_{x_i \in J_u^+} K(\rho_i) \geq K\left(\sum_{x_i \in J_u^+} \rho_i \right)$$

since K is continuous by (K1). By (K2w), we have

$$K(\rho) \leq 2K(\rho/2) \leq \cdots \leq 2^m K(\rho/2^m) = (K(q)/q)\rho$$

for $q = \rho/2^m$. Sending $m \to \infty$, we obtain by (K3) that

$$K(\rho) \leq \rho.$$

We thus observe that

$$TV_K(u) \geq \int_{\Omega\setminus J_u} |Du| + \sum_{x_i\in J_u^+} K(\rho_i) \geq \left(\rho - \sum_{x_i\in J_u^+} \rho_i\right)_+ + K\left(\sum_{x_i\in J_u^+} \rho_i\right)$$
$$\geq K\left(\left(\rho - \sum_{x_i\in J_u^+} \rho_i\right)_+\right) + K\left(\sum_{x_i\in J_u^+} \rho_i\right) \geq K(\rho).$$

The last inequality follows from the subadditivity (K2w). □

If we assume (K1) and (K3), we have, by (2.1),

{ELES} $$K(\rho) > c_M\rho \quad \text{for} \quad 0 < \rho \leq M, \tag{3.1}$$

with some positive constant c_M depending only on M. We next give our monotonicity result.

{TMon}

Theorem 3.4. *Assume that K satisfies (K1), (K2w) and (K3), and that $g \in C[a,b]$. Let U be a minimizer of TV_{Kg} in $BV(a,b)$. Let α and β with $\alpha < \beta$ be in C, where C denotes the coincide set. If $U(\alpha) \leq U(\beta)$ (resp. $U(\alpha) \geq U(\beta)$), then U is non-decreasing (non-increasing) provided that $\beta - \alpha < c_M/(\lambda M)$ if* $\operatorname{osc} g = \max_{[\alpha,\beta]} g - \min_{[\alpha,\beta]} g \leq M$*, where c_M is a constant* (3.1).

Proof. We first note that U is continuous at α, β by Lemma 3.2. Moreover, if there is no point of C in (α, β), by Lemma 3.2, U is piecewise constant with at most one jump in (α, β). Thus, U is monotone.

We next consider the case that $C \cap (\alpha, \beta) \neq \emptyset$. We may assume that $U(\alpha) \leq U(\beta)$ since the proof for the other case is symmetric. We shall prove that

$$U(\alpha) \leq U(x_0) \leq U(\beta)$$

for any $x_0 \in C \cap (\alpha, \beta)$. Suppose that there were a point $x_0 \in C \cap (\alpha, \beta)$ such that $U(\alpha) > U(x_0)$ or $U(\beta) < U(x_0)$. We may assume that $U(x_0) > U(\beta)$ since the proof for the other case is parallel. By Lemma 3.2, we see that there is $x_0 \in C \cap (\alpha, \beta)$ such that

$$U(x_0) = \max_{[\alpha,\beta]} U.$$

We take $x_1 \in C \cap (x_0, \beta]$ such that

$$U(x_1) = \min_{[x_0,\beta]} U,$$

see Figure 4. We now define a truncated function

$$v(x) = \begin{cases} \min\left(U(x), U(x_1)\right), & x \in [\alpha, x_1] \\ U(x), & x \in (x_1, \beta]. \end{cases}$$

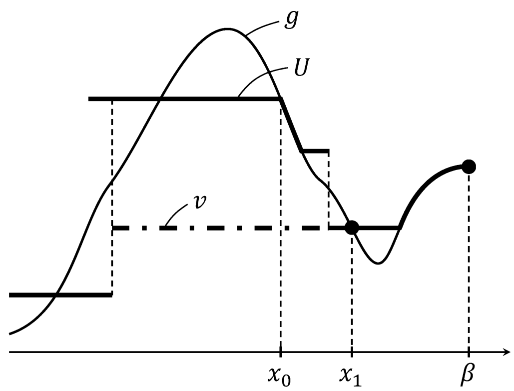


FIGURE 4 FChoi

We shall prove that $TV_{Kg}(v) < TV_{Kg}(U)$ if $\rho = U(x_0) - U(x_1) > 0$. Since U and v are continuous at $x = x_0$ and x_1,

$$TV_K(U) = TV_K(U,(\alpha,x_0)) + TV_K(U,(x_0,x_1)) + TV_K(U,(x_1,\beta)),$$
$$TV_K(v) = TV_K(v,(\alpha,x_0)) + TV_K(v,(x_0,x_1)) + TV_K(v,(x,\beta)).$$

Since TV_K does not increase by truncation, we see

$$TV_K(U,(\alpha,x_0)) \geq TV_K(v,(\alpha,x_0)).$$

Since $v = U$ on (x_1,β), we proceed

$$TV_K(U) - TV_K(v) \geq TV_K(U,(x_0,x_1)) - TV_K(v,(x_0,x_1)) = TV_K(U,(x_0,x_1)).$$

By Lemma 3.3, we observe that

$$TV_K(U,(x_0,x_1)) \geq K(\rho) \quad \text{with} \quad \rho = U(x_0) - U(x_1).$$

Thus by (3.1), we now obtain

$$TV_K(U) - TV_K(v) > c_M\rho.$$

We next compare the values $\mathcal{F}(v)$ and $\mathcal{F}(U)$. By definition, we observe

$$\begin{aligned}\frac{2}{\lambda}(\mathcal{F}(v) - \mathcal{F}(U)) &= \int_\alpha^{x_1} (g-v)^2\,dx - \int_\alpha^{x_1} (g-U)^2\,dx \\ &\leq \int_\alpha^{x_1} \rho\,(|g-v| + |g-U|)\,dx \leq (\beta-\alpha)\rho 2M.\end{aligned}$$

The last inequality follows from the fact that the value of U must be between $\min_{[\alpha,\beta]} g$ and $\max_{[\alpha,\beta]} g$. We thus observe that

$$TV_{Kg}(U) - TV_{Kg}(v) > c_M\rho - \lambda(\beta-\alpha)\rho M = (c_M\rho - \lambda(\beta-\alpha)M)\,\rho.$$

If $\beta-\alpha$ satisfies $\beta-\alpha \leq c_M/(\lambda M)$, U is not a minimizer provided that $\rho > 0$, i.e., $U(x_0) > U(x_1)$. Thus, we conclude that $U(x_0) \leq U(\beta)$.

So far we have proved that U is non-decreasing on C. By Lemma 3.2, we conclude that U itself is non-decreasing in $[\alpha,\beta]$. □

{SGe}

4. Minimiziers for general one-dimensional data

In this section, we shall prove that a minimizer U is piecewise linear if K satisfies (K1), (K3) and (K2) instead of (K2w). In other words, we shall prove our main result.

If we assume (K2), then merging jumps decrease the value TV_K. However, $\mathcal{F}$ may increase. We have to estimate an increase of $\mathcal{F}$.

{SEFid}

4.1. **Bound for an increase of fidelity.** We shall estimate an increase of fidelity $\mathcal{F}$. We begin with a simple setting. We set for $\gamma \in (\alpha,\beta)$,

$$U_0^\gamma(x) = \begin{cases} g(\alpha), & x \in [\alpha,\gamma), \\ g(\beta), & x \in [\gamma,\beta]; \end{cases}$$

see Figure 5. The fidelity of U_0^γ on (α,β) is still denoted by $\mathcal{F}(U_0^\gamma)$, i.e.,

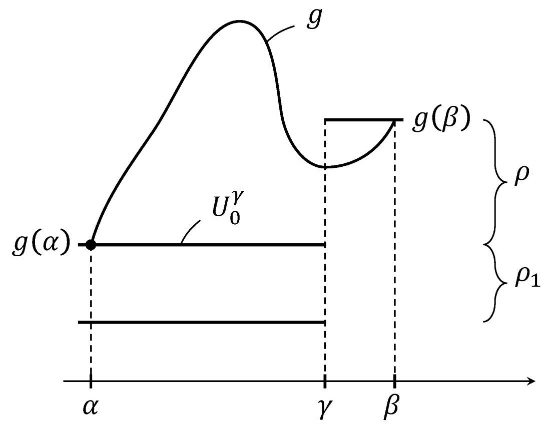


Figure 5. profile of U_0^γ and g FUGamma

$$F(\gamma) = \frac{2}{\lambda}\mathcal{F}\left(U_0^\gamma\right) = \int_\alpha^\beta \left|U_0^\gamma - g\right|^2 dx$$

for $\gamma \in [\alpha,\beta]$. Since we do not assume that g is non-decreasing, g may be very large on (α,γ). Fortunately, we observe that g cannot be too large on (α_0,γ) in the average if $F(\gamma)$ is smaller than $F(\alpha+0)$.

{PCong}

Proposition 4.1. *Assume that $g \in C[\alpha,\beta]$ and $U_0^\gamma(\alpha) = g(\alpha)$, $U_0^\gamma(\beta) = g(\beta)$ with $\rho = U_0^\gamma(\beta) - U_0^\gamma(\alpha) > 0$. If $\gamma \in [\alpha,\beta]$ satisfies $F(\alpha+0) \geq F(\gamma)$, then*

$$\int_\alpha^\gamma g(x)\,dx \leq \frac{1}{2}\left(U_0^\gamma(\alpha) + U_0^\gamma(\beta)\right)(\gamma-\alpha)$$

or

$$\int_\alpha^\gamma \left(g(x) - U_0^\gamma(\alpha)\right) dx \le \rho(\gamma - \alpha)/2.$$

Proof. We observe that

$$\begin{aligned} F(\alpha+0) - F(\gamma) &= \int_\alpha^\gamma \left\{(g(\beta) - g(x))^2 - (g(\alpha) - g(x))^2\right\} dx \\ &= -\int_\alpha^\gamma \rho \{2g - (g(\alpha) + g(\beta))\} \, dx. \end{aligned}$$

Since $\rho > 0$, $F(\alpha+0) - F(\gamma) \ge 0$ implies that

$$\int_\alpha^\gamma 2g \, dx \le \int_\alpha^\gamma (g(\alpha) + g(\beta)) \, dx = \left(U_0^\gamma(\alpha) + U_0^\gamma(\beta)\right)(\gamma - \alpha).$$

□

We give a simple application. See Figure 5.

{LPert}

Lemma 4.2. *Assume the same hypotheses of Proposition 4.1. Then, for* $\rho_1 > 0$*,*

$$\int_\alpha^\gamma \left(U_0^\gamma - \rho_1 - g\right)^2 dx - \int_\alpha^\gamma \left(U_0^\gamma - g\right)^2 dx \le \rho_1(\rho_1 + \rho)(\gamma - \alpha).$$

Proof. We may assume that $U_0^\gamma(\alpha) = 0$ by adding a const ant to both U_0^γ amd g. The left-hand side equals

$$I = \int_\alpha^\gamma (\rho_1 + g)^2 \, dx - \int_\alpha^\gamma g^2 \, dx = \rho_1 \int_\alpha^\gamma \{\rho_1 + 2g\} \, dx.$$

Since

$$2\int_\alpha^\gamma g(x) \, dx \le \rho(\gamma - \alpha)$$

by Proposition 4.1, we end up with $I \le (\rho_1^2 + \rho_1\rho)(\gamma - \alpha)$. □

{RCong}

Remark 4.3. *In Proposition 4.1 and Lemma 4.2, we do not assume that* $g < U_0^\gamma(\beta)$ *on* (γ, β) *nor* $g \ge U_0^\gamma(\alpha)$ *on* (α, γ)*.*

We next consider behavior of g between two points of the coincidence set where U is a constant.

{PFEC}

Proposition 4.4. *Assume that K satisfies (K1), (K2w) and (K3). Assume that $g \in C[a,b]$. Let $U \in BV(a,b)$ by a minimizer of TV_{Kg}. Let $\alpha, \beta \in [a,b)$ be $\alpha < \beta$ and $\alpha, \beta \in C$. Assume that U is non-decreasing and there is $\gamma \in [\alpha, \beta) \cap C$ such that $U(\gamma) = U(\alpha) = g(\alpha)$ and $U(x) > U(\gamma)$ for $x > \gamma$. Assume that there is $p_j \in U(C)$ such that $p_j < U(\beta)$, $p_j \downarrow U(\gamma)$ as $j \to \infty$. Then, $\int_\alpha^\gamma (g(x) - U(\alpha)) \, dx \le 0$.*

Proof. We may assume that $U(\alpha) = g(\alpha) = 0$ so that $\lim_{j\to\infty} p_j = 0$. We set

$$v_j(x) = \begin{cases} \max\left(p_j, U(x)\right), & \alpha_j := \alpha + (\gamma - \alpha)/j < x \\ U(x), & x \le \alpha_j, \end{cases}$$

for $j \ge 2$; see Figure 6. Since U is a minimizer, by definition,

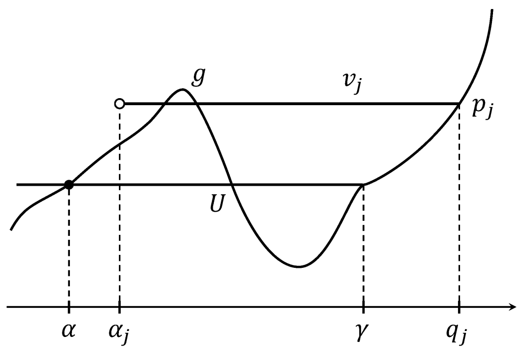


FIGURE 6. v_j and U| FLFEC

$$TV_{Kg}(v_j) \ge TV_{Kg}(U).$$

By Lemma 3.3,

$$\begin{aligned} TV_K(U,(\alpha,\beta)) &\ge TV_K(U,(\alpha,\gamma)) + TV_K\left(U,(\gamma,q_j)\right) + TV_K\left(U,(q_j,\beta)\right) \\ &\ge 0 + K(p_j) + TV_K\left(U,(q_j,\beta)\right) \end{aligned}$$

where $q_j \in U^{-1}(p_j)$. Since

$$TV_K\left(v_j,(\alpha,\beta)\right) = K(p_j) + TV_K\left(U,(q_j,b)\right),$$

$TV_{Kg}(v_j) \ge TV_{Kg}(U)$ implies that $\mathcal{F}(v_j) \ge \mathcal{F}(U)$. In other words,

$$\int_\alpha^{q_j} \left\{(v_j - g)^2 - (U - g)^2\right\} dx \ge 0.$$

Dividing the region of integration (α, q_j) by (α,γ) and (γ, q_j), we obtain

$$\int_{\alpha_j}^{\gamma} \left\{g^2 - (p_j - g)^2\right\} dx \le \int_\gamma^{q_j} \left\{(p_j - g)^2 - (U - g)^2\right\} dx,$$

or

$$p_j \int_{\alpha_j}^{\gamma} (2g - p_j)\, dx \le \int_\gamma^{q_j} (p_j - U)(p_j + U - 2g)\, dx.$$

Since $U \le p_j$ on (γ, q_j), the right-hand side is dominated by

$$p_j \int_\gamma^{q_j} |p_j + U - 2g|\, dx \le p_j |q_j - \gamma| \left(2p_j + 2\|g\|_\infty\right).$$

Thus

$$\int_{\alpha_j}^{\gamma} (2g - p_j)\, dx \leq |q_j - \gamma| \left(2p_j + 2\|g\|_\infty\right).$$

Sending $j \to \infty$ yields that

$$\int_{\alpha}^{\gamma} g\, dx \leq 0,$$

since $q_j \to \gamma$ by our assumption that $U(x) > 0$ for $x > \gamma$ and U is non-decreasing. The proof is now complete. □

We say that a closed interval F is a *facet* of U if F is a maximal nontrivial closed interval such that U is a constant on the interior $\operatorname{int} F$ of F. Let $|F|$ denote its length. We are able to claim a similar statement for each facet of a minimizer U.

{LCGF}

Lemma 4.5. *Assume that K satisfies (K1), (K2w) and (K3). Assume that $g \in C[a,b]$. Let $U \in BV(a,b)$ be a minimizer of TV_{Kg}. Assume that U is non-decreasing. Let $F = [x_0, x_1]$ with $x_1 < b$ be a facet of U. Then*

$$\int_F (g(x) - U)\, dx \leq (U(x_1 + 0) - U(x_1 - 0))\, |F|/2. \tag{4.1}$$ {Ekey}

Proof. We may assume $U \equiv 0$ on F. By Lemma 3.1, $F \cap C \neq \emptyset$. We set

$$\alpha = \inf(F \cap C)$$

which is still in C since C is closed. By Lemma 3.1,

$$g(x) < U(x) = 0 \quad \text{on} \quad [x_0, \alpha)$$

since U is non-increasing. If $U(x_1 + 0) = U(x_1 - 0)$, then $x_1 \in C$ and U is continuous by Lemma 3.2. Thus by Proposition 4.4,

$$\int_{\alpha}^{x_1} g(x)\, dx \leq 0.$$

Thus, we obtain (4.1) when U does not jump at x_1 since we know $g < 0$ on $[x_0, \alpha)$.

If $U(x_1 + 0) - U(x_1 - 0) > 0$, we may apply Proposition 4.1 and conclude that

$$\int_{\alpha}^{x_1} g\, dx \leq U(x_1 + 0)(x_1 - \alpha)/2.$$

Again by $g < 0$ on $[x_0, \alpha)$, we conclude that

$$\int_{\alpha}^{x_1} g\, dx \leq U(x_1 + 0)(x_1 - \alpha)/2 \leq U(x_1 + 0)|F|/2.$$

Since we know that $g < 0$ on $[x_0, \alpha)$, the proof of Lemma 4.5 is now complete. □

{TFidInc}

Theorem 4.6. *Assume that K satisfies (K1), (K2w) and (K3). Assume that $g \in C[a,b]$. Let $U \in BV(a,b)$ be a minimizer of TV_{Kg}. Assume that U is non-decreasing. Let $\alpha, \beta \in C$ with $\alpha < \beta < b$. Then*

$$\int_\alpha^\beta (U(\alpha) - g)^2 \, dx - \int_\alpha^\beta (U - g)^2 \, dx \le \rho^2(\beta - \alpha)$$

when $\rho = U(\beta) - U(\alpha) \ge 0$.

Proof. If $\rho = 0$, a minimizer must be constant $U(\alpha)$ so the above inequality is trivially fulfilled. We may assume $\rho > 0$.

As before, we may assume $U(\alpha) = 0$ so that $U(x) \ge 0$. We proceed

$$\int_\alpha^\beta g^2 \, dx - \int_\alpha^\beta (U - g)^2 \, dx = \int_\alpha^\beta U(2g - U) \, dx.$$

On the coincidence set C,

$$\int_C U(2g - U) \, dx = \int_C U^2$$

since $g = U$ on C. On a facet $F = [x_0, x_1]$, by Lemma 4.5,

$$\begin{aligned}\int_F U(2g - U) \, dx &= U(x_1 - 0) \int_F \{(2g - 2U(x_1 - 0)) + U(x_1 - 0)\} \, dx \\ &\le U(x_1 - 0) \left((U(x_1 + 0) - U(x_1 - 0)) + U(x_1 - 0)\right) |F| \\ &= U(x_1 - 0) \left(U(x_1 + 0) - U(x_1 - 0)\right) |F| \\ &\le \rho^2 |F|\end{aligned}$$

since $x_1 \le \beta < b$. We now conclude that

$$\begin{aligned}\int_\alpha^\beta U(2g - U) \, dx &\le \sum_{i=1}^\infty \int_{F_i} U(2g - U) \, dx + \int_C U(2g - U) \, dx \\ &\le \rho^2 \sum_{i=1}^\infty |F_i| + \int_C U^2 \, dx \le \rho^2 \sum_{i=1}^\infty |F_i| + \rho^2 |C|\end{aligned}$$

Since $(\alpha, \beta) = \bigcup_{i=1}^\infty F_i \cup C$ with at most countably many facets $\{F_i\}$, this implies that

$$\int_\alpha^\beta g^2 \, dx - \int_\alpha^\beta (U - g)^2 \, dx \le \rho^2 \left(\sum_{i=1}^\infty |F_i| + |C| \right) = \rho^2(\beta - \alpha).$$

□

{SSNP}

4.2. **No possibility of fine structure.** Our goal in this subsection is to prove our main theorem. In other words, we shall prove that a minimizer does not allow to have a "fine" structure under the assumption (K2). At the end of this subsolution, we prove our main theorem Theorem 1.1.

{LLEN}

Lemma 4.7. *Assume that K satisfies (K1), (K2) and (K3). Assume that $g \in C[a,b]$. Let $U \in BV(a,b)$ be a minimizer of TV_{Kg}. Assume that U is non-decreasing. Let $\alpha, \beta, \gamma \in C$ satisfy $a \leq \alpha < \gamma < \beta \leq b$ and $U(\beta) - U(\gamma) \geq U(\gamma) - U(\alpha) > 0$. Assume that $(\gamma, \beta) \cap C = \emptyset$. Then*

$$\beta - \alpha > 2C_M/\lambda$$

for $M \geq \operatorname{osc}_{[\alpha,\beta]} g$, where C_M is the constant in (K2).

Proof. We set $\rho_1 = U(\gamma) - U(\alpha)$, $\rho_2 = U(\beta) - U(\gamma)$. Since $(\gamma, \beta) \cap C = \emptyset$, by Lemma 3.2, U has only one jump at $x_1 \in (\gamma, \beta)$ and $U = U(\beta)$ for $x \in (x_1, \beta)$, $U = U(\gamma)$ for $x \in (\gamma, x_1)$.

We set

$$v(x) = \begin{cases} U(\alpha), & x \in (\alpha, x_1), \\ U(x), & x \notin (\alpha, x_1) \end{cases}$$

and compare $TV_K(v)$ with $TV_K(U)$ on (α, β); see Figure 7. Then

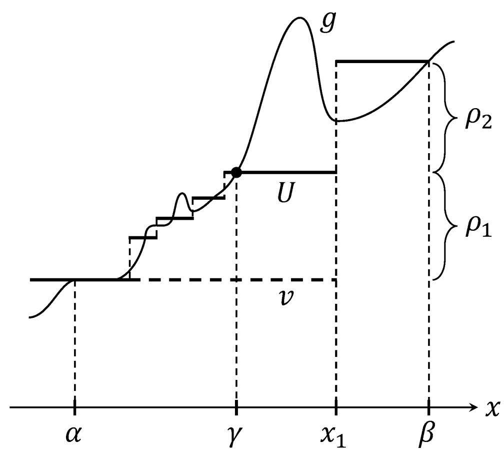


FIGURE 7. the graph of U, g and v FGUV

$$TV_K(U, (\alpha, \beta)) = K(\rho_2) + TV_K(U, (\alpha, \gamma)) \geq K(\rho_2) + K(\rho_1)$$

by Lemma 3.3. Clearly,

$$TV_K(v, (\alpha, \beta)) = K(\rho_1 + \rho_2).$$

By (K2), we see that

$$TV_K(U, (\alpha, \beta)) - TV_K(v, (\alpha, \beta)) \geq K(\rho_1) + K(\rho_2) - K(\rho_1 + \rho_2) \geq C_M \rho_1 \rho_2.$$

Since U minimizes $\int_\gamma^\beta |U-g|^2\,dx$, we have, by Lemma 4.2,

$$\int_\gamma^{x_1} \left\{(U-g)^2 - (v-g)^2\right\} dx \geq -\rho_1(\rho_1+\rho_2)(x_1-\gamma).$$

Since $\gamma < x_1 < b$ and U is a minimizer of TV_{Kg} on (α, x_1), we have, by Theorem 4.6,

$$\int_\alpha^{\gamma} \left\{(U-g)^2 - (v-g)^2\right\} dx \geq -\rho_1^2(\gamma-\alpha).$$

We now conclude that

$$TV_{Kg}(U) - TV_{Kg}(v) \geq C_M\rho_1\rho_2 - \left(\rho_1\rho_2(x_1-\gamma) + \rho_1^2(\gamma-\alpha)\right)\lambda\,/\,2.$$

Since we assume that $\rho_1 \leq \rho_2$, this implies that

$$TV_{Kg}(U) - TV_{Kg}(v) \geq \rho_1\rho_2\left(C_M - (x_1-\alpha)\lambda/2\right).$$

Since U is a minimizer, $C_M - (x_1-\alpha)\lambda/2 \leq 0$, we conclude that

$$\beta - \alpha \geq x_1 - \alpha \geq 2C_M/\lambda.$$

□

From the proof of Lemma 4.7, we have a rather general estimate for U_0^γ defined at the beginning of Section 4.1.

{LGEs}

Lemma 4.8. *Assume that K satisfies (K1), (K2) and (K3). Assume that $g \in C[\alpha,\beta]$ and $M \geq \operatorname{osg}_{[\alpha,\beta]} g$. Assume that $\rho = g(\beta)-g(\alpha) > 0$ and $x_1 \in (\alpha,\beta)$. Let U be a non-decreasing function with $U(\alpha) = g(\alpha)$, $U(\beta) = g(\beta)$ which is continuous at α and β. Assume that $(\alpha,\beta) = \bigcup_{i=1}^\infty F_i \cup C$ where F_i is a facet and C is a coincidence set with $F_1 = [x_1,\beta]$ and $F_2 = [x_0,x_1]$, $x_0 \in (\alpha,x_1)$. (The set F_i for $i \geq 3$ could be empty.) Assume that $U(\beta-0) = g(\beta)$ and $g(\beta) - U(x_1-0) := \rho_2 > \rho/2$. Assume further that U satisfies* (4.1) *on each F_i for $i \geq 3$. Assume that $F(x_1) \leq F(\alpha+0)$ for $U_0^{x_1}$. If C contains an interior point of F_2, then*

$$TV_{Kg}(U) - TV_{Kg}(U_0^{x_1}) \geq \rho_1\rho_2\left(C_M - (x_1-\alpha)\lambda/2\right),$$

with $\rho_1 = \rho - \rho_2$.

We are interested in the case that there are no jumps. We begin with an elementary property of K.

{LEeK}

Lemma 4.9. *Assume that K satisfies (K2) and (K3). Then, $\rho - K(\rho) \geq C_M\rho^2/2$.*

Proof. An iterative use of (K2) yields

$$\begin{aligned} K(\rho) &\le 2K\left(\frac{\rho}{2}\right) - C_M\left(\frac{\rho}{2}\right)^2 \le 2\left(2K\left(\frac{\rho}{4}\right) - C_M\left(\frac{\rho}{4}\right)^2\right) - C_M\left(\frac{\rho}{2}\right)^2 \\ &\le 2^m K\left(\frac{\rho}{2^m}\right) - C_M\rho^2 \sum_{j=1}^{m} 2^{j-1}\left(\frac{1}{2^j}\right)^2 \end{aligned}$$

for $m = 1, 2, \ldots$. Since $\sum_{j=1}^{m} 2^{-j-1} \to 1/2$ as $m \to \infty$, sending $m \to \infty$ yields

$$K(\rho) \le \rho - C_M\rho^2/2$$

by (K3). The proof is now complete. □

{LCont}

Lemma 4.10. *Assume the same hypotheses of Lemma 4.7 concerning K and g. Let $U \in BV(a,b)$ be a minimizer of TV_{Kg}. Assume that U is non-decreasing and continuous on $[\alpha,\beta] \subset [a,b]$ with $\alpha,\beta \in C$. Then $U(\alpha) = U(\beta)$.*

Proof. Suppose that $U(\alpha) \ne U(\beta)$ so that $U(\alpha) < U(\beta)$, there would exist at least one $p \in (U(\alpha), U(\beta))$ such that $U^{-1}(p)$ is a singleton $\{x_0\}$ since U is continuous. By Lemma 3.1, $x_0 \in C$. Moreover, there is a sequence $p_j \downarrow p$ $(j \to \infty)$ such that $U^{-1}(p_j)$ is a singleton $\{x_j\}$. This is possible since the set of values q where $U^{-1}(q)$ is not a singleton is at most a countable set. Since $U^{-1}(p)$ is a singleton, $x_j \downarrow x_0$. Again by Lemma 3.1, $x_j \in C$. We set

$$v_j(x) = \begin{cases} p, & x \in (x_0, x_j), \\ U(x), & x \notin (x_0, x_j). \end{cases}$$

By Theorem 4.6, we obtain

$$\frac{2}{\lambda}\left(\mathcal{F}(v_j) - \mathcal{F}(U)\right) \le \rho_j^2(x_j - x_0)$$

with $\rho_j = p_j - p$.

Since $TV = TV_K$ for a continuous function, we see that

$$TV_K\left(U, (x_0, x_j)\right) = \rho_j.$$

By Lemma 4.9, we observe that

$$TV_K(U) - TV_K(v_j) = \rho_j - K(\rho_j) \ge C_M\rho_j^2/2, \quad M > \operatorname{osc}_{[\alpha,\beta]} g.$$

We thus conclude that

$$TV_{Kg}(U) - TV_{Kg}(v_j) \ge \left(C_M\rho_j^2 - \lambda\rho_j^2(x_j - x_0)\right)/2 = \rho_j^2\left(C_M - \lambda(x_j - x_0)\right)/2.$$

For a sufficiently large j, $C_M - \lambda(x_j - x_0) > 0$ since $x_j \downarrow x_0$. This would contradict to our assumption that U is a minimizer. Thus $U(\alpha) = U(\beta)$. □

{TDis}

Theorem 4.11. *Assume that K satisfies (K1), (K2) and (K3). Assume that $g \in C[a,b]$. Let $U \in BV(a,b)$ be a minimizer of TV_{Kg}. Let $\alpha, \beta \in C$ with $a \le \alpha < \beta \le b$. Assume that $\beta - \alpha \le A_M/\lambda$ with $A_M = \min\{c_M/M, 2C_M\}$ and $\operatorname{osc}_{[\alpha,\beta]} g \le M$, where c_M is in (3.1) and C_M is in (K2). Then U takes either $U(\alpha)$ or $U(\beta)$ on $[\alpha,\beta]$ and it has at most one jump point in (α,β).*

Proof. By Theorem 3.4, U is non-decreasing in $[\alpha,\beta]$. We may assume $U(\alpha) < U(\beta)$. If there is no jump, i.e., $U \in C[\alpha,\beta]$, by Lemma 4.10, U is not a minimizer so U must have at least one jump point in (α,β). We take a jump point x_0 such that jump size

$$U(x_0 + 0) - U(x_0 - 0) \ (> 0)$$

is maximum among all jump size of U in (α,β). If $U(x_0 - 0) = U(\alpha)$, $U(x_0 + 0) = U(\beta)$, we get the conclusion. Suppose that $U(x_0 - 0) > U(\alpha)$. We set

$$\beta' = \inf\left\{x \in (x_0,\beta] \;\middle|\; x \in C\right\},$$
$$\gamma = \sup\left\{x \in (\alpha, x_0] \;\middle|\; x \in C\right\}.$$

By Lemma 3.1, $\beta' > x_0$ and $\gamma < x_0$. Since $U(x_0 - 0) > U(\alpha)$, we see $\gamma > \alpha$. Since the jump at x_0 is a maximal jump, we apply Lemma 4.7 on (α,β') to get

$$\beta' - \alpha > 2C_M/\lambda,$$

which would contradict the assumption $\beta - \alpha < A_M/\lambda$. We thus conclude that $U(x_0 - 0) = U(\alpha)$. If $U(x_0 + 0) < U(\beta)$, we consider $-U(-x)$ instead of U. We argue in the same way. We apply Lemma 4.7 and get a contradiction. We thus conclude that $U(x_0 + 0) = U(\beta)$. The proof is now complete. □

Proof of Theorem 1.1. By Lemma 3.1, $\inf g \le U \le \sup g$ on $[a,b]$ and C is non-empty. We take an integer m so that

$$m > (b - a)\lambda/A_M.$$

We divide $[a,b]$ into m intervals so that the length of each interval is less than A_M/λ and the boundaries of each interval $[x_0, x_1]$ does not contain a jump point of U. (This is possible if we shift x_0, x_1 a little bit unless $x_0 = a'$, or $x_1 = b'$ since J_U is at most a countable set. If $x_0 = a'$ (resp. $x - b'$), U must be continuous at $x_0 = a'$ ($x_1 = b'$) since U is continuous on the coincidence set C by Lemma 3.2.) If $C \cap [x_0, x_1]$ is empty or singleton, then U must be a constant on $[x_0, x_1]$ by Lemma 3.1. We next consider the case that $C \cap [x_0, x_1]$ has at least two points. We set

$$\alpha' = \inf\left(C \cap [x_0, x_1]\right), \quad \beta' = \sup\left(C \cap [x_0, x_1]\right)$$

and may assume $\alpha' < \beta'$. Since $\beta' - \alpha' < A_M/\lambda$, Theorem 4.11 implies that U only takes two values $U(x_0)$ and $U(x_1)$ and has at most one jump on (α', β'). By Lemma 3.1, U is constant on $[x_0, \alpha']$ and $[\beta', x_1]$.

We now observe that on each $[x_0, x_1]$, U has at most one jump. We thus conclude that U is a piecewise constant function with at most m jumps on (a, b). □

{SSMo}

4.3. **Minimizers for monotone data.**

We shall prove that the bound for number of jumps is improved when g is monotone. In other words, we shall prove Theorem 1.2.

We first observe the monotonicity of a minimizer for TV_{Kg} when g is monotone.

{LMon}

Lemma 4.12. *Assume that K satifies (K1) and that $g \in C[a,b]$ is non-decreasing. Then a minimizer of TV_{Kg} (in $BV(\Omega)$) is non-decreasing.*

Proof. Let U be a minimizer. We take its right continuous representation. Suppose that $U(x_0) > U(y_0) \geq g(y_0)$ with some $x_0 < y_0$. Then a chopped function

$$v(x) = \min\left(U(x_0), U(x)\right)$$

decreases both TV_K and the fidelity term so that

$$TV_{Kg}(v) < TV_{Kg}(U);$$

see Figure 8. Thus for $U \geq g$, U is always non-decreasing. A symmetric

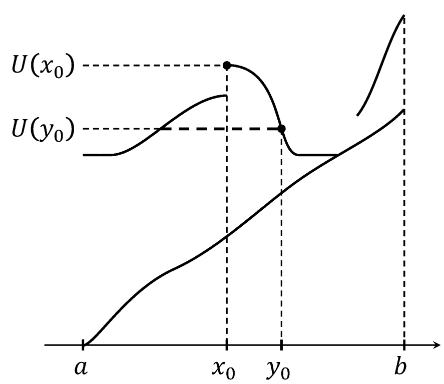


FIGURE 8. chopped function FCh

argument implies that U is always non-decreasing for $U \leq g$. If U is not non-decreasing in (a, b), then it must jump at some x_0 with

$$U(x_0 + 0) \leq g(x_0) \leq U(x_0 - 0), \quad U(x_0 - 0) \neq U(x_0 + 0).$$

Our competitor can be taken as

$$v(x) = \begin{cases} \min\{U(x), g(x_0)\} & \text{for} \quad x < x_0 \\ \max\{U(x), g(x_0)\} & \text{for} \quad x > x_0. \end{cases}$$

By definition, $TV_{Kg}(v) < TV_{Kg}(U)$ so such jump cannot occur for a minimizer. Thus U is non-decreasing. □

Proof of Theorem 1.2. If g is monotone, a minimizer of TV_{Kg} is automatically monotone by Lemma 4.12. We don't need to invoke Theorem 3.4 so the bound c_M/M is unnecessary. Thus Theorem 1.2 follows from Theorem 1.1. □

It is not difficult to get a minimizer when g is strictly increasing for TV_g, i.e.,

$$TV_g(u) = TV(u) + \mathcal{F}(u).$$

By Lemma 4.12, a minimizer U must be non-decreasing. (In this problem, TV_g is strictly convex and lower semicontinuous in $L^2(\Omega)$, so there exists a unique minimizer.) We note that

$$TV(u) = u(b) - u(a)$$

provided that u is non-decreasing. We set $d_1 = u(a) - g(a)$, $d_2 = g(b) - u(b)$ and $a_1 = g^{-1}(u(a))$, $a_2 = g^{-1}(u(b))$. To minimize $\mathcal{F}(u)$, we take d_1 and d_2 such that

$$d_1 = \frac{\lambda}{2} \int_a^{a_1} |u(a) - g|^2 \, dx$$
$$d_2 = \frac{\lambda}{2} \int_{a_2}^{a} |u(b) - g|^2 \, dx.$$

See Figure 9. Then the minimizer U must be

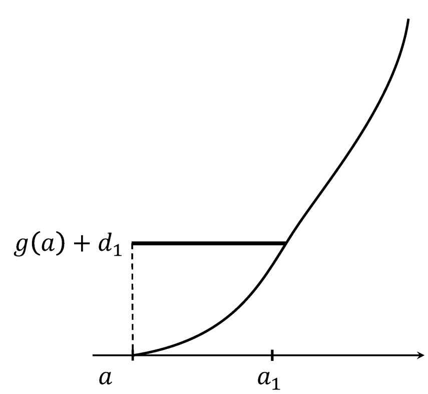


FIGURE 9. near the boundary FB

$$U(x) = \min\{\max(g(a) + d_1, u(x)), g(b) - d_2\}$$

provided that $g(a) + d_1 \le g(b) - d_2$. This formation of a flat part near the boundary occurs by the natural boundary condition. (In general, the minimizer has no jumps if g is continuous for TV_g (cf. [CL], [GKL]).)

In the case when g is monotone, we are able to prove a conclusion stronger than Proposition 4.1.

{PConMo}

Proposition 4.13. *Assume that $g \in C[\alpha,\beta]$ and $U_0^\gamma(\alpha) = g(\alpha)$, $U_0^\gamma(\beta) = g(\beta)$ with $\rho = U(\beta) - U(\alpha) > 0$. If $\gamma \in [\alpha,\beta]$ satisfies $F(\alpha + 0) \ge F(\gamma)$, then $g(x) - U_0^\gamma(\alpha) \le \rho/2$ for $x \in [\alpha,\gamma]$.*

This easily follows from the next observation.

{PMF}

Proposition 4.14. *Assume that $g \in C[\alpha,\beta]$ is non-decreasing. Then $F(\gamma)$ is minimized if and only if $g(\gamma) = (g(\alpha) + g(\beta))/2$.*

Proof. A direct calculation shows that

$$\begin{aligned} F(\gamma) &= \int_\alpha^\beta |U_0^\gamma - g|^2\, dx \\ &= \int_\alpha^\gamma |g(x) - g(\alpha)|^2\, dx + \int_\gamma^\beta |g(\beta) - g(x)|^2\, dx. \end{aligned}$$

Then

$$\begin{aligned} F'(\gamma) &= |g(\gamma) - g(\alpha)|^2 - |g(\beta) - g(\gamma)|^2 \\ &= (g(\alpha) + g(\alpha) - 2g(\gamma))(g(\beta) - g(\alpha)). \end{aligned}$$

Thus, F takes its only minimum at $(g(\alpha) + g(\beta))/2$. □

We give a few explicit estimate of TV_{Kg} for monotone function g to give an alternate proof for Theorem 1.2.

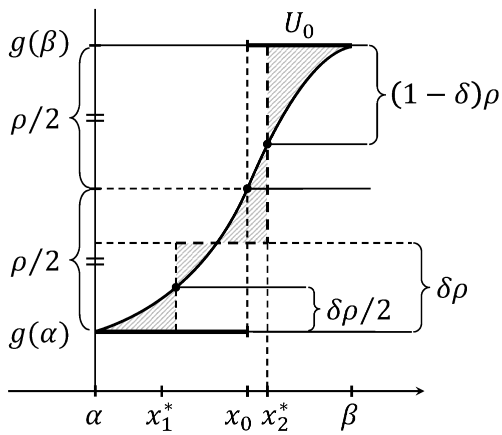
We recall U_0^γ. We shall simply write U_0^γ by U_0 if $g(\gamma) = (g(\alpha) + g(\beta))/2$. Note that such γ always exists since g is continuous. It is unique if g is increasing. In general, it is not unique and we choose one of such γ. We next consider a non-decreasing piecewise constant function with two jumps. We set $\rho = g(\beta) - g(\alpha)$ and $\delta \in (0,1)$ and define

$$U_\delta(x) = \begin{cases} g(\alpha), & x \in [\alpha, x_1) \\ g(\alpha) + \delta\rho, & x \in [x_1, x_2) \\ g(\beta), & x \in [x_2, \beta) \end{cases}$$

for $x_1 < x_2$ with $x_1, x_2 \in (\alpha,\beta)$. If $\mathcal{F}(U_\delta)$ is minimized by moving x_1 and x_2 by Lemma 3.1, there is a point $y_0 \in (x_1, x_2)$ such that $U_\delta(y_0) = g(\alpha) + \delta\rho$. Moreover, by Proposition 4.14, $\mathcal{F}(U_\delta)$ is minimized by taking $x_1 = x_1^*$, $x_2 = x_2^*$, where x_1^* and x_2^* are defined by

$$g(x_1^*) = g(\alpha) + \delta\rho/2, \quad g(x_2^*) = (g(\beta) + g(\alpha) + \delta\rho)/2 = g(\beta) - (1-\delta)\rho/2,$$

see Figure 10. Again x_1^*, x_2^* may not be unique. We take one of them.



FUD

Figure 10. U_0 and U_δ

We shall estimate

$$TV_{Kg}(U_\delta) - TV_{Kg}(U_0).$$

This can be considered as a special case of Lemma 4.8 since (4.1) is automatically fulfilled by Proposition 4.13. However, the proof is clearer than that of general g, we give a full proof.

{LFid}

Lemma 4.15. *Assume that $g \in C[\alpha,\beta]$ is increasing. Let v be a non-decreasing piecewise constant function with three values $g(\alpha)$, $g(\alpha)+\delta\rho$, $g(\beta)$ for $\delta \in (0,1)$, where $\rho = g(\beta) - g(\alpha)$. Then*

$$\int_\alpha^\beta (U_0 - g)^2 \, dx - \int_\alpha^\beta (v-g)^2 \, dx \le \delta(1-\delta)\rho^2(\beta-\alpha).$$

Proof. We may assume $\delta \le 1/2$ by symmetry. For U_δ, we fix $x_1 = x_1^*$ and $x_2 = x_2^*$. We first observe that $x_1^* < x_0 < x_2^*$. We calculate

$$\int_\alpha^\beta |U_\delta - g|^2 dx = \int_\alpha^{x_1^*} |g(x) - g(\alpha)|^2 \, dx + \int_{x_1^*}^{x_2^*} |g(x) - g(\alpha) - \delta\rho|^2 \, dx$$
$$+ \int_{x_2^*}^\beta |g(\beta) - g(x)|^2 \, dx.$$

We shall estimate the second term of the right-hand side by dividing

$$\int_{x_1^*}^{x_2^*} = \int_{x_1^*}^{x_0} + \int_{x_0}^{x_2^*} = I_1 + I_2.$$

For I_1, we proceed

$$I_1 - \int_{x_1^*}^{x_0} |g(x) - g(\alpha)|^2 \, dx = \int_{x_1^*}^{x_0} -\delta\rho \left(2g(x) - 2g(\alpha) - \delta\rho\right) dx$$
$$\geq -\delta(1-\delta)\rho^2 (x_0 - x_1^*)$$

since $g(x) - g(\alpha) \leq \rho/2$ by $\delta < 1/2$ on (x_1^*, x_0) so that $2g(x) - 2g(\alpha) - \delta\rho \leq (1-\delta)\rho$. Since

$$I_2 = \int_{x_0}^{x_2^*} |g(x) - g(\beta) - (1-\delta)\rho|^2 \, dx,$$

as for I_1, we have

$$I_2 - \int_{x_0}^{x_2^*} |g(\beta) - g(x)|^2 \, dx \geq -\delta(1-\delta)\rho^2 (x_2^* - x_0).$$

We thus conclude that

$$\int_\alpha^\beta |U_\delta - g|^2 dx - \int_\alpha^\beta |U_0 - g|^2 dx \geq -\delta(1-\delta)\rho^2 (x_2^* - x_1^*).$$

This yields the desired inequality for v since $\mathcal{F}(v) \geq \mathcal{F}(U_\delta)$. □

We set $\rho = g(\beta) - g(\alpha)$ and

$$X_\delta = \Big\{ v \;\Big|\; v \text{ is a non-decreasing piecewise constant function with three facets and } v(\alpha) = g(\alpha), v(\beta) = g(\beta). \text{ Moreover, the value on the middle facet equals } g(\alpha) + \delta\rho \Big\}.$$

{LND}

Lemma 4.16. *Assume that K satisfies (K1) and (K2). Assume that $g \in C[\alpha, \beta]$ is non-decreasing. If $g(\beta) - g(\alpha) \leq M$ and $C_* := C_M - (\beta - \alpha)\lambda/2 > 0$, then*

$$\inf_{v \in X_\delta} TV_{Kg}(v) \geq TV_{Kg}(U_0) + C_* \delta(1-\delta)\rho^2$$

for $\delta \in (0,1)$, where $\rho = g(\beta) - g(\alpha)$.

Proof. Since $TV_K(U_\delta)$ is independent of the choice of x_1, x_2, we observe that

$$\inf_{v \in X_\delta} TV_{Kg}(v) \geq TV_{Kg}(U_\delta)$$

with $x_1 = x_1^*$, $x_2 = x_2^*$. For $\rho = g(\beta) - g(\alpha)$, we set $\rho_1 = \delta\rho$, $\rho_2 = (1-\delta)\rho$ so that $\rho_1 + \rho_2 = \rho$. We observe that

$$TV_K(U_\delta) = K(\rho_1) + K(\rho_2), \quad TV_K(U_0) = K(\rho).$$

By (K2), we observe that

$$TV_K(U_\delta) \geq TV_K(U_0) + C_M \rho_1 \rho_2$$

for $\rho \le M$. By Lemma 4.15, we conclude that

$$\begin{aligned} TV_{Kg}(U_\delta) &= TV_K(U_\delta) + \mathcal{F}(U_\delta) \\ &\ge TV_K(U_0) + C_M\rho_1\rho_2 + \mathcal{F}(U_0) - \frac{\lambda}{2}\delta(1-\delta)\rho^2(\beta-\alpha) \\ &= TV_{Kg}(U_0) + \delta(1-\delta)\rho^2\left(C_M - (\beta-\alpha)\lambda/2\right). \end{aligned}$$

Thus

$$TV_{Kg}(U_\delta) - TV_{Kg}(U_0) \ge C_*\delta(1-\delta)\rho^2$$

provided that $(\beta-\alpha)\lambda/2 < C_M$, i.e., $C_* > 0$. The proof is now complete. □

To prove Theorem 1.2 directly, we approximate a general function by a piecewise constant function such that TV_K is also approximated.

{LAp1}

Lemma 4.17. *Assume that K satisfies (K1), (K3) and that $\Omega = (a,b)$. For any $u \in (L^p \cap BV)(\Omega)$ with $p \ge 1$, there is a sequence of piecewise constant functions $\{u_m\}$ (with finitely many jumps) such that $u_m \to u$ in $L^p(\Omega)$ and $TV_K(u_m) \to TV_K(u)$ as $m \to \infty$.*

Since $u_m \to u$ in $L^2(\Omega)$ implies $\mathcal{F}(u_m) \to \mathcal{F}(u)$ when $g \in L^2(\Omega)$, this lemma yields

{LAp2}

Lemma 4.18. *Assume that K satisfies (K1), (K3) and $\Omega = (a,b)$. For any $u \in (L^2 \cap BV)(\Omega)$, there is a sequence of piecewise constant function $\{u_m\}$ such that $u_m \to u$ in $L^2(\Omega)$ and $TV_{Kg}(u_m) \to TV_{Kg}(u)$ as $m \to \infty$.*

We shall prove Lemma 4.17 by reducing the problem when u is continuous, i.e., $u \in C[a,b]$.

{LApC}

Lemma 4.19. *Assume that K satisfies (K3) and $\Omega = (a,b)$. For any $u \in C[a,b] \cap BV(\Omega)$, there is a sequence of piecewise constant functions $\{u_m\}$ such that $u_m \to u$ in $C[a,b]$ and $TV_K(u_m) \to TV_K(u)$ as $m \to \infty$.*

Proof. For a continuous function u, $TV_K(u)$ agrees with usual $TV(u)$. We extend u continuously in some neighborhood if $\overline{\Omega}$ and denote it by $\overline{u}$. We mollify $\overline{u}$ by a symmetric mollifier ρ_ε. It is well known that $u_\varepsilon = \overline{u} * \rho_\varepsilon$ is C^∞ in $[a,b]$ and $u_\varepsilon \to u$ in $C[a,b]$ as $\varepsilon \to 0$. Moreover, $TV(u_\varepsilon) \to TV(u)$ [Giu, Proposition 1.15]. Since ρ_ε can be approximated (in C^1 sense) by polynomials in a bounded set, we approximate u_ε by a polynomial with its derivative in $C[a,b]$. Thus, we may assume that u is a polynomial.

For a given $\eta > 0$, we define a piecewise constant function

$$u^\eta(x) = k\eta \quad \text{if} \quad k\eta \le u(x) < (k+1)\eta.$$

We divide the interval (a,b) into finitely many subintervals $\{(a_i, a_{i+1})\}_{i=0}^{\ell}$ with $a = a_0 < a_1 < \cdots < a_\ell < a_{\ell+1} = b$ such that on each such an interval u is either increasing or decreasing. This is possible since u is a polynomial. Let $TV(u,(a_i,a_{i+1}))$ denote the total variation of u in (a_i,a_{i+1}) for $i=0,\dots,\ell$. Then

$$0 \leq TV(u,(a_i,a_{i+1})) - TV(u^\eta,(a_i,a_{i+1})) \leq 2\eta.$$

By the assumption (K3), we see that for any $\delta>0$ there is $\eta_0>0$ such that $|\eta-K(\eta)|<\delta\eta$ for $\eta<\eta_0$. Since u is continuous, the size of jumps of u^η is always η so

$$|TV(u^\eta,(a_i,a_{i+1})) - TV_K(u^\eta,(a_i,a_{i+1}))| \leq \delta TV\left(u_\eta,(a_i,a_{i+1})\right) \quad \text{for} \quad \eta<\eta_0,$$

where we use the same convention to TV_K. We thus observe that

$$\begin{aligned}|TV(u)-TV_K(u^\eta)| &\leq \sum_{i=0}^{\ell}\left(2\eta+\delta TV(u^\eta,(a_i,a_{i+1}))\right)\\ &= 2\eta(\ell+1)+\delta TV(u^\eta) \leq 2\eta(\ell+1)+\delta TV(u).\end{aligned}$$

Sending $\eta\to 0$, we now conclude that

$$\lim_{\eta\to 0}|TV(u)-TV_K(u^\eta)| \leq \delta TV(u).$$

Since $\delta>0$ is arbitrary, the convergence $TV_K(u^\eta)\to TV(u)$ as $\eta\to 0$ follows. By definition, $u^\eta\to u$ in $C[a,b]$. The proof is now complete. □

Proof of Lemma 4.17. Since u is bounded by $u\in BV(\Omega)$, for any $\delta>0$, the set J_δ of jump discontinuities of u whose jump size greater than δ is a finite set. For any $\varepsilon>0$, we take $\delta>0$ such that

$$\sum_{x\in J_u\setminus J_\delta} K\left(|u^+-u^-|(x)\right) < \varepsilon.$$

We may assume that $J_\delta=\{a_j\}_{j=1}^{\ell}$ with $a_j<a_{j+1}$ and $a_0=a$, $a_{\ell+1}=b$. In each interval (a_j,a_{j+1}), we approximate u in L^p with continuous function u_ε. We now apply Lemma 4.19 on each interval (a_j,a_{j+1}) to approximate u_ε by a piecewise constant function u_ε^η. Although the jump at a_j is not exactly equal to

$$|u^+-u^-|(a_j),$$

it converges to this value as $\eta\to 0$ and $\varepsilon\to 0$. We now obtain a desired sequence of piecewise constant functions. □

{RApMon}

Remark 4.20. *If u is non-decreasing, it is rather clear that u_m in Lemmas 4.17, 4.18, 4.19 can be taken as a non-decreasing function by construction.*

We give a sufficient condition (Lemma 4.16) that a larger jump costs less for TV_{Kg} not only TV_K by giving a quantative estimate.

In the rest of this section, we give an alternate proof of Theorem 1.2 without using Theorem 1.1 by establishing a quantative estimate of TV_{Kg}. By approximation (Lemma 4.18), if there is a minimizer of TV_{Kg} among piecewise constant functions, this minimizer is also a minimizer of TV_{Kg} in BV. The existence of a minimizer among piecewise constant functions can be proved since the number of facets are restricted. We shall come back to this point at the end of this section. Unfortunately, this argument is not enough to prove Theorem 1.2 since we do not know the uniqueness of a minimizer. To show Theorem 1.2, we need a quantified version of Lemma 4.16. Let U_0 be a minimizer with one jump.

{LEBJ}

Lemma 4.21. *Assume that K satisfies (K1), (K2) and (K3). Assume that $g \in C[\alpha,\beta]$ is non-decreasing. Let M be a constant such that $\rho = g(\beta) - g(\alpha) \le M$. Assume that $C_* = C_M - \lambda(\beta-\alpha)/2 > 0$. Let v is a non-decreasing function on $[\alpha,\beta]$ which is continuous at α and β and $v(\alpha) = g(\alpha)$, $v(\beta) = g(\beta)$. Then*

$$TV_{Kg}(v) \ge \frac{C_*}{2}\left(\left(\sum_{i=1}^{\infty}\rho_i\right)^2 - \sum_{i=1}^{\infty}\rho_i^2 + \left(\rho - \sum_{i=1}^{\infty}\rho_i\right)^2\right) + TV_{Kg}(U_0),$$

where $\rho_i = v(z_i+0) - v(z_i-0) > 0$ and $J_v = \{z_i\}_{i=1}^{\infty}$ $(\subset (\alpha,\beta))$ is the set of jump discontinuities. The non-negative term

$$\left(\sum_{i=1}^{\infty}\rho_i\right)^2 - \sum_{i=1}^{\infty}\rho_i^2 + \left(\rho - \sum_{i=1}^{\infty}\rho_i\right)^2$$

vanishes if and only if $\rho = \rho_{i_0}$ with some i_0.

We begin with an elementary lemma on a sequence.

{LSeq}

Lemma 4.22. *Let $\{m_k\}_{k=1}^{\infty}$ be an increasing sequence of natural numbers and $\lim_{k\to\infty} m_k = \infty$. Let $\{\rho_i^k\}_{1\le i\le m_k}$ be a set of non-negative real numbers. Assume that*

$$s := \lim_{k\to\infty} s_k > 0 \quad \textit{for} \quad s_k = \sum_{i=1}^{m_k}\rho_i^k.$$

(i)

$$\lim_{k\to\infty}\sum_{1\le i,j\le m_k}\rho_i^k\rho_j^k = s^2/2$$

provided that $\lim_{i\to\infty}\rho_i^k = 0$.

(ii)

$$\varliminf_{k\to\infty}\sum_{1\le i,j\le m_k}\rho_i^k\rho_j^k \ge \sum_{1\le i,j\le m_k}\rho_i\rho_j$$

provided that $\lim_{i\to\infty}\rho_i^k = \rho_i \ge 0$.

(iii) *Let I be the set of i such that $\rho_i = 0$ and $i \notin I$ implies $\rho_i > 0$. Then*

$$\lim_{k\to\infty} \sum_{1\le i,j\le m_k} \rho_i^k \rho_j^k \ge \sum_{1\le i,j\le m_k} \rho_i\rho_j + s_{I^c}^2/2 \quad \textit{for} \quad s = s_I + s_{I^c}, \quad s_I = \sum_{i=1}^{\infty} \rho_i$$

$$= \left(s_I^2 - \sum_{i=1}^{\infty} (\rho_i)^2 + s_{I^c}^2 \right) \Big/ 2.$$

Proof. (i) We may assume that $\rho_1^k \ge \rho_2^k \ge \cdots$. Since

$$\sum_{1\le i,j\le m_k} \rho_i^k \rho_j^k = \left(s_k^2 - \sum_{i=1}^{m_k} (\rho_i^k)^2 \right) \Big/ 2$$

and

$$\sum_{i=1}^{\infty} (\rho_i^k)^2 \le \rho_1^k \sum_{i=1}^{\infty} \rho_i^k = \rho_1^k s_k \to 0 \quad \text{as} \quad k \to \infty,$$

we obtain (i).

(ii) This follows from Fatou's lemma.

(iii) We divide the set of indices by I and I^c, the complement of I. Since

$$\sum_{1\le i,j\le m_k} \rho_i^k \rho_j^k \ge \sum_{\substack{1\le i<j\le m_k \\ i,j\in I}} \rho_i^k \rho_j^k + \sum_{\substack{1\le i,j\le m_k \\ i,j\in I^c}} \rho_i^k \rho_j^k,$$

applying the results (i), (ii) yield (iii).

□

Proof of Lemma 4.21. We may assume that $g(\beta) > g(\alpha)$. Let U be a minimizer of TV_{Kg} on BV. Its existence is proved in Section 2. By Lemma 4.12, U is non-decreasing. Moreover, by Lemma 3.2, U is a piecewise constant outside the set

$$C = \left\{ x \in \Omega \;\middle|\; U(x) = g(x) \right\}$$

and each facet contains a point z such that $U(z) = g(z)$ and U is continuous at z. We set

$$a = \inf C, \quad b = \sup C.$$

By definition, $a \ge \alpha$ and $b \le \beta$. By Lemma 4.12 and Lemma 3.2, $g(\alpha) \le U \le g(\beta)$ and U is continuous at a and b. We approximate U on $[a, b]$ by a piecewise constant non-decreasing function u_k with $m_k + 1$ facets by Lemma 4.19. Since U is continuous at a and b, we may assume that $u_k(x) = g(\alpha)$ for $x > a$ close to a and $u_k(x) = g(\beta)$ for $x < b$ close to b. We denote jumps by $a_1 < a_2 < \cdots < a_{m-1}$ with $m = m_k + 1$ and set $h_i = u_k(a_i + 0)$ with convention that $a_0 = a$, $a_m = b$. We set

$$\rho_i = h_i - h_{i-1} \quad \text{for} \quad i = 1, \ldots, m_k$$

which denotes the jump at each a_i. We fix ρ_i and minimize TV_{Kg} on (a, b). In other words, we minimize $\mathcal{F}$ by moving a_i's. Let $\bar{u}_k$ be its minimizer. By Proposition 4.14, facets of $\bar{u}_k$ consist of

$$[a, x_1], [x_1, x_2], \dots, [x_{m_k-1}, b]$$

with $g(a_i) = (g(x_i) + g(x_{i+1}))/2$, $i = 1, \dots, m_k - 1$; see Figure 11 with $x_0 = a_0, \dots, x_{m-1} = a_m = b$. By this choice,

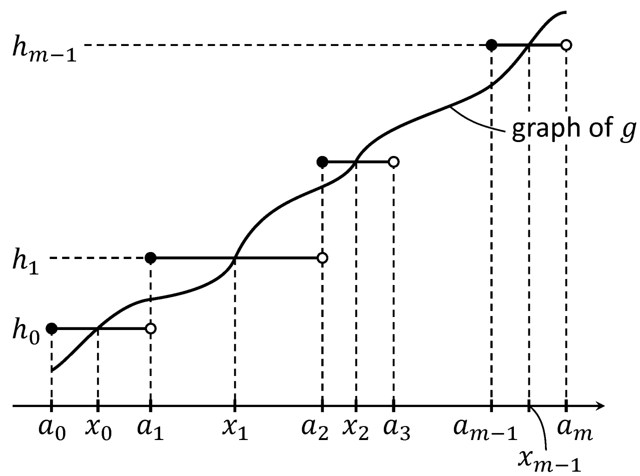


Figure 11. graphs of g and $U\lceil^{\mathtt{FPrU}}$

$$TV_{Kg}(u_k) \geq TV_{Kg}(\bar{u}_k).$$

We shall estimate $TV_{Kg}(\bar{u}_k)$ from below as in Lemma 4.16. For (a, x_i), let V_i be a piecewise constant function on $[a, x_i]$ with one jump at the point y_i such that

$$g(y_i) = (g(x_i) + g(a))\,/\,2 \quad \text{and} \quad V_i(a) = g(\alpha), \quad V_i(x_i) = u_k(x_i).$$

We set

$$W_i(x) = \begin{cases} V_i(x), & x \in [a, x_i) \\ \bar{u}_k(x), & x \in [x_i, b]. \end{cases}$$

We note that $W_{m_k} = V_{m_k} = U_0$. Since $C_* > 0$, we argue as in Lemma 4.16 and observe that

$$\begin{aligned} TV_{Kg}(\bar{u}_k) &\geq TV_{Kg}(W_2) + C_*\rho_1\rho_2 \\ &\geq TV_{Kg}(W_3) + C_*\rho_3(\rho_1 + \rho_2) + C_*\rho_1\rho_2 \\ &\dots \\ &\geq TV_{Kg}(V_{m_k}) + C_* \sum_{1 \leq i,j \leq m_k} \rho_i\rho_j = TV_{Kg}(U_0) + C_* \sum_{1 \leq i,j \leq m_k} \rho_i\rho_j. \end{aligned}$$

From now on, we write jumps of u_k by ρ_i^k instead of ρ_i. Our estimate for $TV_K(\bar{u}_k)$ yields

$$TV_{Kg}(u_k) \geq TV_{Kg}(U_0) + C_* \sum_{1 \leq i,j \leq m_k} \rho_i^k \rho_j^k.$$

Since $u_k \to U$ in L^2 and $TV_{Kg}(u_k) \to TV_{Kg}(U)$, we observe, by changing numbering of i in ρ_i^k if necessary, that $\rho_i^k \to \rho_i > 0$ as $k \to 0$ for i such that ρ_i corresponds to jumps at z_i of U and $\rho_i^k \to 0$ for other i's. The convergence of TV_{Kg} prevents that two jumps with positive length merges at the limit unless one of them tends to zero. Let I be the set such that $\rho_i^k \to \rho_i > 0$. We now apply Lemma 4.22 to conclude that

$$\begin{aligned} TV_{Kg}(v) \geq TV_{Kg}(U) &\geq TV_{Kg}(U_0) + \varliminf_{k\to\infty} C_* \sum_{1 \leq i,j \leq m_k} \rho_i^k \rho_j^k \\ &\geq TV_{Kg}(U_0) + \frac{C_*}{2} \left(s_I^2 - \sum_{i=1}^{\infty} \rho_i^2 + s_{I^c}^2 \right), \end{aligned}$$

where $s_I = \sum_{i=1}^{\infty} \rho_i \leq g(\beta) - g(\alpha)$. Since

$$s_I^2 - \sum_{i=1}^{\infty} \rho_i^2 \geq 0 \quad \text{and} \quad s_{I^c}^2 \geq 0,$$

the quantity

$$s_I^2 - \sum_{i=1}^{\infty} \rho_i^2 + s_{I^c}^2 = 0$$

if and only if $s_{I^c}^c = 0$ and $\left(\sum_{i=1}^{\infty} \rho_i\right)^2 = \sum_{i=1}^{\infty} \rho_i^2$, i.e., I is a singleton, i.e., $I = \{i_0\}$ with some $i_0 \in I$ and $\rho_{i_0} = \rho$. □

We conclude this subsection by proving Theorem 1.2 based on the quantative estimate.

Alternate proof of Theorem 1.2. Again, we may assume that g is not a constant. By Lemma 4.12, a minimizer U is non-decreasing. By Lemma 3.2, at the place where $C = \{U(x) = g(x)\}$, the function U is continuous. Let $Q \subset [g(a), g(b)]$ be the set of q such that

$$I_q = \left\{ x \mid q(x) = q \right\}, \quad I_q^* = \left\{ x \in I_q \mid x > \inf I_q \right\}$$

are not a singleton. We set

$$C_\Gamma = C \setminus \bigcup_{q \in Q} I_q^*.$$

Since g is continuous, C_Γ is not empty by Lemma 3.1.

If we prove that C_Γ is a discrete set so that it is a finite set. Then, by Lemma 3.1, U is a piecewise constant function. We shall prove that C_Γ is a discrete set. By definition, if $x_1 < x_2$ for $x_1, x_2 \in C_\Gamma$, then $g(x_1) < g(x_2)$. Suppose that C_Γ were not discrete, then for any small ε, there would exist $x_1, x_2 \in C_\Gamma$ with $x_1 < x_2 < x_1 + \varepsilon$ such that the interval (x_1, x_2) would contain infinitely many element of C_Γ. Since U minimizes TV_{Kg} on $(x_1 x_2)$ with $U(x_1) = g(x_1)$, $U(x_2) = g(x_2)$, applying Lemma 4.21 to $\alpha = x_1, \beta = x_2$ to conclude that U must have at most one jump provided that $\varepsilon < 2C_M/\lambda$. This yields a contradiction so we conclude that C_Γ is discrete and U is a non-decreasing piecewise constant function. The number of jumps can be estimated since the distance of two points in C is at most $2C_M/\lambda$. □

4.4. **Application of approximation lemma.** We shall prove Theorem 1.3. We approximate $g \in L^\infty(a,b)$ by $g_\ell \in C[a,b]$ such that $g_\ell \to g$ in $L^2(a,b)$ as $\ell \to \infty$ and

$$\operatorname{ess.inf} g \le g_\ell \le \operatorname{ess.sup} g \quad \text{on} \quad (a,b)$$

for all $\ell \ge 1$. Let $u \in BV(a,b)$ with $TV_{Kg}(u) < \infty$. By Lemma 4.17, there is a sequence $\{u_\ell\}$ of piecewise constant functions such that $TV_K(u_\ell) \to TV_K(U)$ with $u_\ell \to u$ in $L^2(a,b)$ as $\ell \to \infty$. We thus observe that $\{u_\ell\}$ is a "recovery" sequence in the sense that $TV_{Kg_\ell}(u_\ell) \to TV_{Kg}(u)$. Let U_ℓ be a minimizer of TV_{Kg_ℓ}. By Theorem 1.1, the number of jump m_ℓ is bounded by

$$m_\ell \le \left[(b-a)\lambda \,/\, (2C_M)\right] + 1 = m_*$$

with $M = \operatorname{osc}_{[a,b]} g \ge \operatorname{osc}_{[a,b]} g_\ell$. By compactness of a bounded set in a finite dimensional space, U_ℓ has a convergent subsequence, i.e., $U_\ell \to U$ with some U in $L^2(a,b)$ (and also with respect to weak* topology of BV). Moreover, the limit U is still a piecewise constant function with at most m_* jumps. By lower semicontinuity of TV_K (Proposition 2.4), we see that

$$TV_{Kg}(U) \le \varliminf_{\ell\to\infty} TV_{Kg_\ell}(U_\ell) \le \varliminf_{\ell\to\infty} TV_{Kg}(u_\ell) = TV_{Kg}(u).$$

We thus conclude that U is a desired minimizer of TV_{Kg} since $u \in BV(a,b)$ is an arbitrary element.

5. A SUFFICIENT CONDITION FOR (K2)

{SKdef}

We give a sufficient condition for (K2) if K is derived as a limit of the Kobayashi-Warren-Carter energy, i.e., K is of the form (1.7).

We first consider a kind of Fenchel dual of a function f. We set

{EHdef}

$$H(\rho) = \inf_{x>0} (\rho x + f(x)) \tag{5.1}$$

for a real-valued function f on $[0,\infty)$. If we use the Fenchel dual, it can be written as

$$H(\rho) = -f^*(-\rho) = -\sup_x \Big((-\rho)x - \bar{f}(x)\Big),$$

where $\bar{f}(x) = f(x)$ for $x \geq 0$ and $\bar{f}(x) = \infty$ for $x < 0$. We assume

(f1) $f \in C^1(0,1] \cap C[0,1]$;
(f2) f takes the only minimum 0 at $x = 1$. In other words, $f(x) \geq 0$ for all $x \geq 0$ and $f(x) = 0$ if and only if $x = 1$;
(f3) $f'(x) < 0$ for $x \in (0,1)$.

{LK2}

Lemma 5.1. *Assume (f2). Then $H(\rho) > 0$ for $\rho > 0$ and $H(0) = 0$. Assume further (f2) and (f3). Then $H(\rho) < f(0)$ for all $\rho > 0$ and there is $x_\rho \in (0,1)$ for $\rho > 0$ such that $H(\rho) = \rho x_\rho + f(x_\rho)$ and $H(\rho)$ is strictly increasing in ρ. Moreover, x_ρ is strictly increasing and $x_\rho \uparrow 1$ as $\rho \downarrow 0$. Furthermore, it satisfies (K2) with $K = H$ if f satisfies*

$$\varliminf_{\rho \downarrow 0} \left(f \circ (-f')^{-1} \right)(\rho) \,/\, \rho^2 > 0. \tag{5.2}$$ {EHC}

Here we define

$$(-f')^{-1}(\rho) = \min \left\{ x \in [0,1] \;\middle|\; f'(x) = -\rho \right\}.$$

Proof. The positivity for $H(\rho)$ for $\rho > 0$ and $H(0) = 0$ is clear by the definition (5.1). Also, the existence in $[0,\eta]$ of a minimizer easily follows by (f1) and (f2). Although x_ρ may not be unique, $x_\rho \uparrow 1$ as well as monotonicity of x_ρ is guaranteed by (f3). The bound $H(\rho) < f(0)$ is rather clear.

It remains to prove that (5.2) yields (K2). Since x_ρ is the minimizer, it must satisfy

$$\rho = (-f')(x_\rho).$$

We take $x_\rho = (-f')^{-1}(\rho)$ in Lemma 5.1. Since

$$H(\rho) = \rho x_\rho + f(x_\rho)$$

by definition, we observe that for $\delta \in (0,1)$

$$\begin{aligned} H(\delta\rho) - \delta H(\rho) &\geq \rho_1 x_{\rho_1} + f(x_{\rho_1}) - \left(\delta\rho x_{\rho_1} + \delta f(x_{\rho_1}) \right) \text{ with } \rho_1 = \delta\rho \\ &= (1-\delta) f(x_{\rho_1}). \end{aligned} \tag{5.3}$$ {EEH}

For $\rho_2 = (1-\delta)\rho$, we have

$$H\left((1-\delta)\rho\right) - (1-\delta)H(\rho) \geq \delta f(x_{\rho_2}).$$

We thus observe that

$$H(\rho_1) + H(\rho_2) - H(\rho) \geq (1-\delta) f(x_{\rho_1}) + \delta f(x_{\rho_2}).$$

By (5.2), we may assume that

$$\left(f \circ (-f')^{-1} \right)(\rho) \geq C_M \rho^2$$

with some $C_M > 0$ provided that $0 \le \rho \le M$. Thus

$$\begin{aligned}(1-\delta)f(x_{\rho_1}) &= (1-\delta)\left(f\circ(-f')^{-1}\right)(\rho_1) \ge C_M(1-\delta)(\delta\rho)^2,\\ \delta f(x_{\rho_1}) &\ge C_M\delta\left((1-\delta)\rho\right)^2.\end{aligned}$$

We now observe that

$$\begin{aligned}H(\rho_1)+H(\rho_2)-H(\rho) &\ge C_M\left((1-\delta)\delta\rho^2\delta+(1-\delta)\delta\rho^2(1-\delta)\right)\\ &= C_M\left((1-\delta)\delta\rho^2\right) = C_M\rho_1\rho_2.\end{aligned}$$

We have proved (K2) for H. □

{LK3}

Lemma 5.2. *Assume that (f1), (f2) and (f3). Then* $\lim_{\rho\downarrow 0} H(\rho)/\rho = 0$. *In other words, H satisfies (K3) with $K = H$.*

Proof. Taking $x = 1$ in (5.1), we see that $H(\rho) \le \rho$. We observe that

$$\begin{aligned}H(\rho)-\rho &= \min_{x>0}\left(\rho(x-1)+f(x)\right)\\ &= \rho\left((x_\rho-1)+f(x_\rho)/\rho\right) \text{ or}\\ \frac{H(\rho)}{\rho}-1 &= x_\rho - 1 + \frac{f(x_\rho)}{\rho}.\end{aligned}$$

Since $f \ge 0$ and $x_\rho \to 1$ as $\rho \downarrow 0$, we conclude

$$\varliminf_{\rho\downarrow 0}\left(\frac{H(\rho)}{\rho}-1\right) \ge 0+0,$$

which now yields (K3). □

{RKCon}

Remark 5.3. (i) *Without* (5.2) *we only get (K2w) since $f \ge 0$ and the estimate* (5.3).

(ii) *If $f(x) = |x-1|^m$ for $m > 0$, then* (5.2) *holds if and only if $m \ge 2$. In fact, $f'(x) = m|x-1|^{m-2}(x-1)$ so that $(-f')^{-1}(\rho) = 1-(\rho/m)^{1/(m-1)}$. The function $\left(f\circ(-f')^{-1}\right)(\rho) = (\rho/m)^{m/(m-1)}$ so* (5.2) *holds if and only if $m \ge 2$.*

(iii) *We may take other element of a preimage of $-f'$ of ρ but as a sufficient condition the present choice is the weakest assumption.*

We come back to (1.7). In other words,

$$K(\rho) = \min_{\xi}\left(\rho(\xi_+)^2 + 2G(\xi)\right), \quad G(\xi) = \left|\int_1^{\xi}\sqrt{F(\tau)}\,d\tau\right|.$$

We assume that

(F1) $F \in C[0,\infty)$ and F takes the only minimum 0 at $x = 1$.

If we set $f(x) = 2G(x^{1/2})$, the property (F1) implies (f1), (f2) and (f3). Since (5.2) is property near $x = 1$, (5.2) for f and G are equivalent. We thus obtain an sufficient condition so that K in (1.7) satisfies (K1), (K2) and (K3)

{PSuff}

Proposition 5.4. *Assume (F1) and*

$$\varliminf_{\rho\downarrow 0} \left(G \circ (-G')^{-1} \right)(\rho) \,/\, \rho^2 > 0. \tag{5.4}$$ {EKSu}

Then K in (1.7) *satisfies (K1), (K2) and (K3).*

Proof. By Lemma 5.1, conditions (K1), (K2) are fulfilled. The property (K3) follows from Lemma 5.2. □

We conclude this section by examining the property (5.4). The left-hand side is equivalent to saying that

$$\varliminf_{\rho\downarrow 0} \int_{F^{-1}(\rho^2)}^{1} \sqrt{F(\tau)}\, d\tau \,/\, \rho^2 > 0,$$

where $F^{-1}(\rho^2) = \min\left\{ x \in [0,1] \,\middle|\, F(x) = \rho^2 \right\}$. We set

$$\bar{F}(x) = F(1-x).$$

The condition (5.4) is now equivalent to

$$\varliminf_{\rho\downarrow 0} \int_{0}^{\bar{F}^{-1}(\rho^2)} \sqrt{\bar{F}(\tau)}\, d\tau \,/\, \rho^2 > 0, \tag{5.5}$$ {EFban}

where $\bar{F}^{-1}(y) = \max\left\{ x \in (0,1) \,\middle|\, \bar{F}(x) = y \right\}$. To simplify the argument, we further assume that

(F2) $F' < 0$ in $(0,1)$ so that the inverse function F^{-1} in $(0, F(0))$ is uniquely determined.

If we assume (F1) and (F2), by changing the variable of integration $\tau = \bar{F}^{-1}(s^2)$, we have

$$\int_0^{\bar{F}^{-1}(\rho^2)} \sqrt{\bar{F}(\tau)}\, d\tau = \int_0^{\rho^2} s \frac{d\tau}{ds}\, ds = \int_0^{\rho^2} 2s^2 (\bar{F}^{-1})'(s^2)\, ds.$$

The condition (5.5) is fulfilled if

$$\varliminf_{\sigma\downarrow 0} \sigma (\bar{F}^{-1})'(\sigma) > 0$$

or equivalently

$$\varlimsup_{\eta\downarrow 0} \bar{F}'(\eta) \Big/ \eta < \infty.$$

We thus obtain a simple sufficient condition.

{TSuff}

Theorem 5.5. *Assume that (F1) and (F2). Then K in* (1.7) *satisfies (K1), (K2), (K3) provided that*

$$\varlimsup_{x\uparrow 1} F'(x)/(x-1) < \infty.$$

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